Acton-Boxborough Math Competition 2026 Solutions

ABMC Team

April 11, 2026

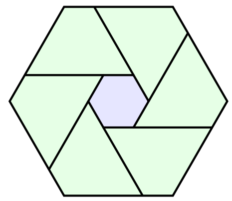
Speed Round

1. **Problem:** What is $2+0+2+6$?

   Solution: $2+0+2+6 = \boxed{10}$.

   Proposed by Anirudh Pulugurtha, Solution by Jonathan Ren

2. **Problem:** In the following figure, the small blue hexagon has an area of 1, and the largest surrounding hexagon has an area of 13. The 6 green trapezoids are all congruent. What is the area of one of the green trapezoids?

   Solution: Note that the area of the large surrounding hexagon equals the area of the small blue hexagon combined with the area of the green region. Therefore, the total area of the six trapezoids equals the area of the large hexagon minus that of the small hexagon, which is $13 - 1 = 12$.

   So the area of each individual trapezoid is simply the green area divided amongst each of the six trapezoids, which equals $12/6 = \boxed{2}$.

   Remark: While it is true that the larger and smaller hexagon must be regular, this piece of information is not needed to solve the problem.

   Proposed by Eric Shi Chen, Solution by Jonathan Ren

3. **Problem:** If the radius of a circle increases by 20%, by what percentage does its area increase?

   Solution: Let the radius be r. If the radius is increased by 20%, then the new radius is equal to $1.2r$. Furthermore, if the radius of the circle is r, then the area of the circle is equal to πr^2.

   Therefore, the new area of the circle would be $\pi(1.2r)^2 = 1.44\pi r^2$. This is a $\frac{1.44-1}{1} \cdot 100\% = \boxed{44}$ percent increase in area.

   Proposed by Eric Shi Chen, Solution by Eric Li

4. **Problem:** Justin rolls a pair of fair dice. The probability that the sum of rolls is less than or equal to 4 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

   Solution: There are a total of $6 \cdot 6 = 36$ different ways to roll the two dice. Of these possibilities, only 6 of them have a sum that is less than or equal to 4, namely $(1,1), (1,2), (1,3), (2,1), (2,2)$, and $(3,1)$. Therefore, the probability that the sum of rolls is less than or equal to 4 is $\frac{6}{36} = \frac{1}{6}$.

   Finally, we see that $a = 1$ and $b = 6$ and thus $a + b = \boxed{7}$.

   Proposed by Eric Shi Chen, Solution by Eric Li

5. **Problem:** How many integers from 1 to 15, inclusive, have the same remainder when divided by 2 as when divided by 3?

   Solution: The remainders repeat in blocks of $2 \cdot 3 = 6$, so we only need to find the remainders of the first 6 integers:

integer	**1**	2	3	4	5	**6**
remainder when divided by 2	**1**	0	1	0	1	**0**
remainder when divided by 3	**1**	2	0	1	2	**0**

Notice that only 1 and 6 have the same remainder when divided by 2 as when divided by 3. Therefore, we see that $1 + 6 = 7$, $6 + 6 = 12$, and $7 + 6 = 13$ also have the same remainder when divided by 2 as when divided by 3. Hence, we see that only $1, 6, 7, 12$, and 13 satisfy the criteria for a total of $\boxed{5}$ integers.

Proposed by Anirudh Pulugurtha, Solution by Eric Li

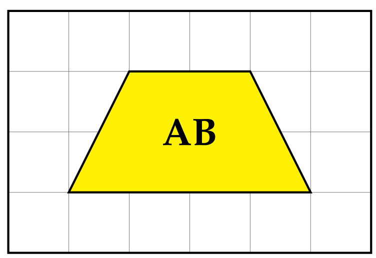
6. **Problem:** Bob buys gumballs after school to eat as he walks home. The gumball machine he visits contains 5 cherry, 3 lemon, 7 green apple, and 2 grape gumballs. When he inserts a coin, a random gumball drops out. How many coins must he insert into the machine to guarantee that he gets at least 3 differently flavored gumballs?

   Solution: To guarantee that Bob gets at least 3 flavors, we need to completely exhaust the two flavors with the highest quantities first, which are the 7 green apple gumballs and 5 cherry gumballs. Once we do this, the next gumball must have a different flavor than green apple or cherry, which means that Bob will get a third flavor on his next coin.

   Therefore, Bob will need to insert $5 + 7 + 1 = \boxed{13}$ coins to guarantee that he gets at least 3 differently flavored gumballs.

   Proposed by Jonathan Ren and Eric Shi Chen, Solution by Eric Li

7. **Problem:** In the diagram below, a point is randomly selected. The probability that it is also within the yellow region "AB" can be written as $\frac{x}{y}$, where x and y are relatively prime positive integers. What is $x + y$?



   Note: The point selected can be anywhere inside the 4 by 6 rectangle shown, but it cannot land outside of the rectangle.

   Solution: The area of the rectangle is $4 \cdot 6 = 24$. The yellow region is a trapezoid with bases of length 2 and 4 and a height of 2. We find that the area of the trapezoid is $\frac{2+4}{2} \cdot 2 = 6$, so the probability that a randomly selected point is inside the 4 by 6 rectangle is inside the yellow region is $\frac{6}{24} = \frac{1}{4}$, so $x + y = 1 + 4 = \boxed{5}$.

   Proposed by Aarush Kulkarni, Solution by Eric Shi Chen

8. **Problem:** In ABMC problem writing, one hard problem takes 1 minute to write. It is also known that:

   (a) 3 easy problems take the same amount of time to write as a medium problem

   (b) 2 medium problems take the same amount of time to write as a hard problem

   (c) 7 hard problems take the same amount of time to write as a sadistic problem

   How many seconds would it take to write one easy problem, two medium problems, three hard problems, and four sadistic problems?

   Solution: Writing 1 easy problem would take the equivalent amount of time as writing

$$\textbf{1 easy problem} \cdot \frac{\textbf{1 medium problem}}{\textbf{3 easy problems}} \cdot \frac{\textbf{1 hard problem}}{\textbf{2 medium problems}} = \frac{1}{6}\ \textbf{hard problems}$$

Similarly, writing 2 medium problem would take the equivalent amount of time as writing

$$\textbf{2 medium problems} \cdot \frac{\textbf{1 hard problem}}{\textbf{2 medium problems}} = \textbf{1 hard problem}$$

Finally, writing 4 sadistic problems would take the equivalent amount of time as writing

$$\textbf{4 sadistic problems} \cdot \frac{\textbf{7 hard problems}}{\textbf{1 sadistic problem}} = \textbf{28 hard problems}$$

Therefore, writing one easy problem, two medium problems, three hard problems, and four sadistic problems would take the same amount of time as writing $\frac{1}{6} + 1 + 3 + 28 = \frac{193}{6}$ hard problems. Each problem takes 1 minute = 60 seconds to write, so writing $\frac{193}{6}$ hard problems would take $\frac{193}{6} \cdot 60 = \boxed{1930}$ seconds.

Proposed by Anirudh Pulugurtha and Eric Shi Chen, Solution by Eric Li

9. **Problem:** Ryan and Tommy are dueling in their favorite video game. Tommy has a skill level of 1, while Ryan has a skill level of 100. When they duel, the person with the greater skill level wins. Every time Ryan wins, Ryan will go easier on Tommy and halve his own skill level, while Tommy improves his skill by one level. How many games will they play until Tommy wins his first game (including the game Tommy wins)?

   Solution: Let T denote Tommy's skill level and R denote Ryan's skill level.

   In round 1, we see that $T = 1$ and $R = 100$. Since $R > T$, we see that Ryan wins.

   In round 2, we see that $T = 2$ and $R = 50$. Since $R > T$, we see that Ryan wins.

   In round 3, we see that $T = 3$ and $R = 25$. Since $R > T$, we see that Ryan wins.

   In round 4, we see that $T = 4$ and $R = 12.5$. Since $R > T$, we see that Ryan wins.

   In round 5, we see that $T = 5$ and $R = 6.25$. Since $R > T$, we see that Ryan wins.

   In round 6, we see that $T = 6$ and $R = 3.125$. Since $R < T$, we see that Tommy wins. Therefore, Tommy will win after round $\boxed{6}$.

   Proposed by Eric Shi Chen and Jonathan Ren, Solution by Eric Li

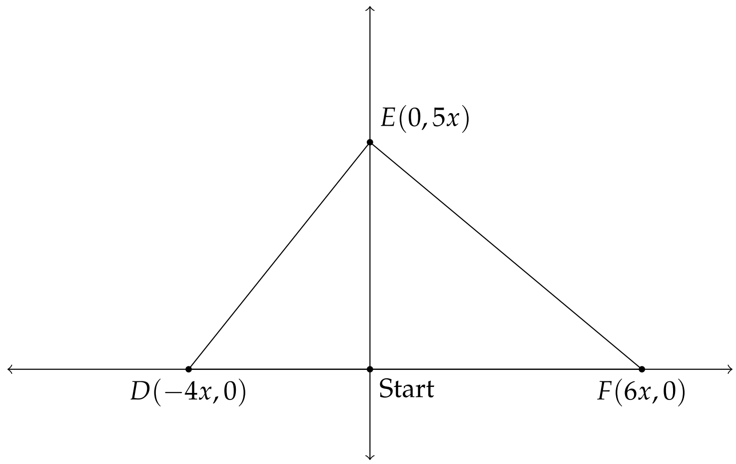
10. **Problem:** There are 10 players on a basketball team, and 5 players have to be chosen for the starting lineup, where the order of the 5 players chosen does not matter. How many ways are there to choose the starting lineup?

    Solution: Since we are picking 5 people out of 10 and the order of the chosen players does not matter, the number of ways to pick these 5 players is $\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \boxed{252}$.

    Proposed by Eric Shi Chen, Solution by Eric Li

11. **Problem:** Eric, Derek, and Fred start running from the same location. Eric runs west at 4 miles per hour, Derek runs north at 5 miles per hour, and Fred runs east at 6 miles per hour. After x hours, Eric is at point E, Derek is at D, and Fred is at F. Find the value of x such that the area of $\triangle DEF$ is 100 square miles.

    Solution:



Note that when you travel at a rate r miles per hour for x hours, you travel a distance of $r \cdot x$ miles. Observe from the diagram that the DEF is a triangle with base of length $10x$ and height of length $5x$. The area of the triangle is thus

$$\frac{1}{2}(10x)(5x) = 25x^2.$$

Since we have that the area is 100, we have that $x^2 = 4$ and $x = \boxed{2}$.

Proposed by Eric Shi Chen, Solution by Anirudh Pulugurtha

12. **Problem:** Richard has 35 balls and stacks them into a triangular pyramid. How many balls are in the 2nd layer from the bottom?

    Solution: Notice that the n-th layer of the pyramid contains $\frac{n(n+1)}{2}$ balls (which is the n-th triangular number).

    We will now calculate the amount of balls present in each layer:

    The 1st layer must consist of 1 ball.

    The 2nd layer would have 3 balls, for a total of $1 + 3 = 4$ balls used.

    The 3rd layer would have 6 balls for a total of $4 + 6 = 10$ balls used.

    The 4th layer would have 10 balls for a total of $10 + 10 = 20$ balls used.

    The 5th (and final) layer would have 15 balls for a total of $20 + 15 = 35$ total balls used in the pyramid.

    Since the fifth layer is the bottom-most layer, the second layer from the bottom would be the fourth layer, which has $\boxed{10}$ balls.

    Proposed by Ryan Xia, Solution by Eric Li

13. **Problem:** Define an *interesting* number as any number that is composed of only the digits 6 and 7. Find the units digit of the sum of the squares of all 4 digit interesting numbers.

    Solution: Every one of the four digits can either be a 6 or a 7, so there are $2^4 = 16$ possible four digit numbers. Of these 16 possible numbers, exactly half end in 6 and the other half end in 7. Squaring these, we get 36 and 49 respectively. So, we get $8(6 + 9) = 120$ which makes the answer $\boxed{0}$.

    Proposed by Jonathan Ren, Solution by Benjamin Li

14. **Problem:** Find the sum of all solutions to the equation $(x^2 + 3x + 1)^{(x^2 - 17x + 60)} = 1$.

    Solution: Recall that, for any real number k, $1^k = 1$ and $k^0 = 1$. Therefore, $(x^2 + 3x + 1)^{(x^2 - 17x + 60)}$ equals 1 either when $x^2 + 3x + 1 = 1$, $x^2 - 17x + 60 = 0$, or $x^2 + 3x + 1 = -1$ when $x^2 - 17x + 60$ is even. Solving each individual equation for x,

$$\begin{aligned} x^2+3x+1 &= 1 \\ x(x+3) &= 0 \\ x &= -3, 0. \\ x^2+3x+1 &= -1 \\ (x+1)(x+2) &= 0 \\ x &= -1, -2. \\ x^2-17x+60 &= 0 \\ (x-12)(x-5) &= 0 \\ x &= 5, 12. \end{aligned}$$

Therefore, the sum of all solutions is $-3+0-1-2+5+12 = \boxed{11}$.

Remark: There is contention over whether the value 0^0 should be equal to $0, 1$, or is indeterminate, and no answer is generally ccepted as "correct". However, this is irrelevant to this problem, as no solution for x causes 0^0 to occur.

Proposed by Nathan Tan, Solution by Jonathan Ren

15. **Problem:** There are 2026 coins on a table, and each coin is labeled with an integer. The sum of the values of these coins is 22286. A mysterious machine does the following operation:

    (a) Pick any two coins. Call the numbers on these coins a and b.

    (b) Replace the coins with new coins valued at $\frac{a+69b}{2}$ and $\frac{a-67b}{2}$.

    After the machine performs this operation 2048 times, what is the sum of the numbers of the 2026 coins?

    Solution: Note that the sum of the numbers on the coins is invariant under the operation: suppose the machine selects two coins a and b, we have that the sum of two new coins is

    $$\frac{a+69b}{2} + \frac{a-67b}{2} = a+b.$$

    Thus, our answer is $\boxed{22286}$.

    Proposed by Aarush Kulkarni and Eric Shi Chen, Solution by Aarush Kulkarni

16. **Problem:** Find the smallest positive integer $n > 2019$ such that $2^{2020} + 2^{2025} + 2^{2026} + 2^{2027} + 2^n$ is a perfect square.

    Solution: Since we are given that $n > 2019$, we can start by letting $n = 2020 + k$, where k is a nonnegative integer. Then we can factor the expression:

    $$2^{2020} + 2^{2025} + 2^{2026} + 2^{2027} + 2^n = 2^{2020}(1 + 32 + 64 + 128 + 2^k) = 2^{2020}(225 + 2^k).$$

    Since 2^{2020} is a perfect square, $225 + 2^k$ must be a perfect square for the entire expression to be a perfect square. We can start checking for nonnegative integers values of k in increasing order, and we find that when $k = 6$, $225 + 2^k = 289 = 17^2$. Therefore, our answer is $2020 + k = \boxed{2026}$.

    Remark: Even if a positive integer n is less than or equal to 2019, the expression cannot be a perfect square. We can prove this as follows:

Assume by contradiction there exists $n \leq 2019$ such that $2^{2020} + 2^{2025} + 2^{2026} + 2^{2027} + 2^n$ is a perfect square. Then, we can rewrite $n = 2020 - m$ where $m \geq 1$, and then we can factor the expression:

$$2^{2020} + 2^{2025} + 2^{2026} + 2^{2027} + 2^n = 2^{2020}(225 + 2^{n-2020}) = 2^n(1 + 225 \cdot 2^m).$$

If n is odd then $1 + 225 \cdot 2^m$ must be divisible by a positive odd power of 2 in order for the expression to be a perfect square, but this is impossible since $1 + 225 \cdot 2^m$ is odd. So, n must be even. This implies that $1 + 225 \cdot 2^m$ must be a perfect square. Suppose $1 + 225 \cdot 2^m = k^2$, where k is a positive integer. Then, we can factor:

$$\begin{aligned} 1 + 225 \cdot 2^m &= k^2 \\ k^2 - 1 &= 225 \cdot 2^m \\ (k+1)(k-1) &= 225 \cdot 2^m \end{aligned}$$

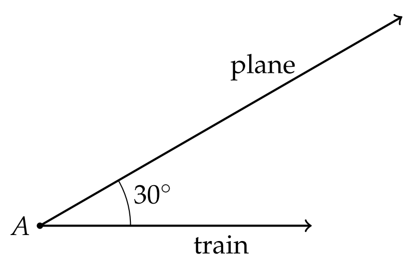
Because $225 \cdot 2^m$ is even, k must be odd, so we can let $k - 1 = 2s$ and $k + 1 = 2s + 2$. Then our expression turns into

$$(s)(s+1) = 225 \cdot 2^{m-2}.$$

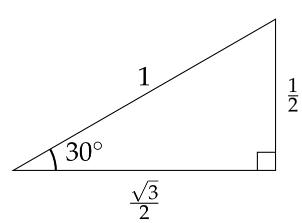
However, since s and $s + 1$ have a different parity, we must have one of them be even and one of them be odd, so $\{s, s+1\} = \{225, 2^{m-2}\}$ This is impossible since neither 224 nor 226 are powers of 2. Since our assumption leads to an impossible result, there exists no positive integers $n \leq 2019$ such that the expression is a perfect square.

Proposed by Eric Huang, Solution by Eric Shi Chen and Benjamin Li

17. **Problem:** At noon, an airplane takes off from point A, maintaining a $30°$ angle with the ground and traveling north. One hour later, a train leaves point A at the same speed and direction as the plane, but parallel to the ground. The time in hours after noon that it will take for the plane to be exactly above the train can be written as $a + b\sqrt{c}$, where a, b, and c are positive integers and $b\sqrt{c}$ is fully simplified. What is $a + b + c$?



Solution: Let point A be the origin of the coordinate plane, and let the train travel in the positive x axis direction. Suppose both the airplane and the train are traveling at k units per hour, and let t be the time in hours after noon.



Consider the airplane's path in the diagram above. For every 1 distance the plane travels physically (the hypotenuse of the right triangle), its x coordinate changes by $\frac{\sqrt{3}}{2}$. So, after t hours, the airplane's x coordinate is $\frac{\sqrt{3}}{2}kt$. The train starts 1 hour after noon, so after t hours, the train's x coordinate is

$k(t-1)$. When the airplane is directly above the train, their x coordinates are equal, so we can set these two expressions as equal and solve:

$$\begin{aligned}\frac{\sqrt{3}}{2}kt &= k(t-1)\\ \frac{\sqrt{3}}{2}t &= t-1\\ t(2-\sqrt{3}) &= 2\\ t &= \frac{2}{2-\sqrt{3}} = 4+2\sqrt{3}.\end{aligned}$$

Thus, our answer is $4+2+3=\boxed{9}$.

Proposed by Jonathan Ren, Solution by Eric Shi Chen and Benjamin Li

18. **Problem:** Let p and q be the roots of the equation $x^2-111x+2026=0$. If $\frac{a}{b}=(1+\frac{1}{p}+\frac{1}{p^2}+\cdots)(1+\frac{1}{q}+\frac{1}{q^2}+\cdots)$, where a and b are relatively prime positive integers, find $a+b$.

    Solution: First, note that the two products $(1+\frac{1}{p}+\frac{1}{p^2}+\cdots)$ and $(1+\frac{1}{q}+\frac{1}{q^2}+\cdots)$ are infinite geometric series with ratios $\frac{1}{p}$ and $\frac{1}{q}$. Therefore, the product can be simplified using the converging geometric series formula as

$$\frac{1}{1-\frac{1}{p}}\cdot\frac{1}{1-\frac{1}{q}} = \frac{1}{\frac{p-1}{p}}\cdot\frac{1}{\frac{q-1}{q}} = \frac{pq}{(p-1)(q-1)}.$$

    Now, the denominator can also be written as $(p-1)(q-1)=pq-p-q+1=pq-(p+q)+1$. By Vieta's formulas

$$pq=2026 \quad p+q=111.$$

    This means that the numerator is 2026 and then the denominator is

$$2026-111+1=1916.$$

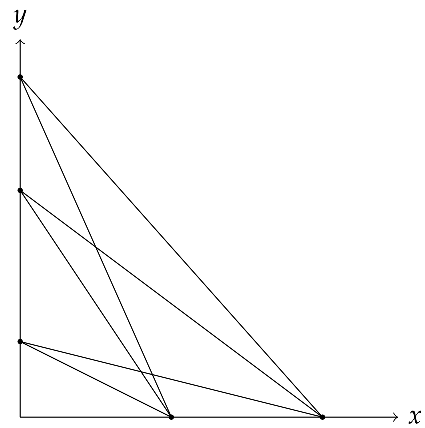
    Therefore,

$$\frac{a}{b}=\frac{2026}{1916}=\frac{1013}{958}\Rightarrow a+b=\boxed{1971}.$$

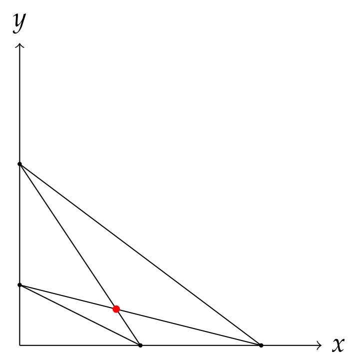
    Proposed by Eric Xiang, Solution by Nathan Tan

19. **Problem:** There are 10 distinct points on the positive x-axis and 20 distinct points on the positive y-axis, and a line segment is drawn from each point on the positive x-axis to each point on the positive y-axis, making 200 total segments. Given that no three segments are concurrent (except on the axes), find the number of intersection points of the segments with a positive x and y coordinate.

    An example with 3 points on the y -axis and 2 points on the x-axis is shown below:



Solution: Note that for any 2 points chosen on the x-axis and 2 points chosen on the y-axis, there is exactly one intersection:



Since no three segments are concurrent, the number of intersections equal to the number of ways there are to choose 2 points on the x-axis and 2 points on the y-axis. The number of ways to choose 2 points on the x-axis is $\binom{10}{2}$ and the number of ways to choose 2 points on the x-axis is $\binom{20}{2}$. Therefore, the total number of intersections is

$$\binom{10}{2}\binom{20}{2} = \boxed{8550}.$$

Proposed by Eric Shi Chen, Solution by Nathan Tan

20. **Problem:** Let $\triangle A_0B_0C_0$ be a triangle with side lengths $A_0B_0 = 1, A_0C_0 = 6$, and $B_0C_0 = \sqrt{37}$. For integers $n \geq 1$, let $\triangle A_nB_nC_n \sim \triangle A_{n-1}B_{n-1}C_{n-1}$ such that $B_nC_n = A_{n-1}C_{n-1}$. What is the sum of the areas of all triangles $A_nB_nC_n$ for $n = 0, 1, 2, \ldots$?

Solution: Note that $\triangle A_0B_0C_0$ is a right triangle of area 3. The main idea is that the ratio of the areas of consecutive triangles is constant, or that the sequence of areas forms an arithmetic series. To show this, it suffices to compute the common ratio between the areas of $\triangle A_nB_nC_n$ and $\triangle A_{n-1}B_{n-1}C_{n-1}$: in particular, for the n^{th} triangle, the similarity $\triangle A_nB_nC_n \sim \triangle A_{n-1}B_{n-1}C_{n-1}$ implies that

$$\frac{B_nC_n}{B_{n-1}C_{n-1}} = \frac{A_{n-1}C_{n-1}}{B_{n-1}C_{n-1}}.$$

Since the triangles are all similar to $\triangle A_0B_0C_0$, we obtain that

$$\frac{A_{n-1}C_{n-1}}{B_{n-1}C_{n-1}} = \frac{A_0C_0}{B_0C_0} = \frac{6}{\sqrt{37}},$$

whence it is clear that the area of $\triangle A_nB_nC_n$ is exactly $\frac{36}{37}$ times the area of $\triangle A_{n-1}B_{n-1}C_{n-1}$. Thus, the sum of the areas of all triangles $A_nB_nC_n$ for $n \geq 0$ is exactly

$$\begin{aligned} 3 + 3\left(\frac{36}{37}\right) + 3\left(\frac{36}{37}\right)^2 + \ldots &= \frac{3}{1 - 36/37} \\ &= 3(37) \\ &= \boxed{111}. \end{aligned}$$

Proposed by Tanish Parida, Solution by Aarush Kulkarni

21. **Problem:** There is exactly one ordered pair of positive integers (a, b) such that $a^2 - 4b^2 = a - 2b + 2^{2026}$, and the ordered pair can be expressed as $(v^w + x, y^z)$ for positive integers v, w, x, y, and z such that v and y are minimized. Compute $v + w + x + y + z$.

    Solution: We can factor the left-hand side as:

$$\begin{aligned} (a - 2b)(a + 2b) &= a - 2b + 2^{2026} \\ (a - 2b)(a + 2b - 1) &= 2^{2026} \end{aligned}$$

    Notice that exactly one of $a - 2b$ and $a + 2b - 1$ is odd because they differ by $4b - 1$.

    Since the only odd factor of 2^{2026} is 1, and $a - 2b < a + 2b - 1$ (since a and b are positive), we know that $a - 2b = 1$ and $a + 2b - 1 = 2^{2026}$.

    We can now solve for a and b:

$$\begin{aligned} a - 2b &= 1 \\ a + 2b - 1 &= 2^{2026} \\ 2a &= 2^{2026} + 2 \\ a &= 2^{2025} + 1 \\ 2^{2025} + 1 - 2b &= 1 \\ b &= 2^{2024} \end{aligned}$$

    Therefore, $v = 2$, $w = 2025$, $x = 1$, $y = 2$, and $z = 2024$. So, $v + w + x + y + z = \boxed{4054}$.

    Proposed by Eric Shi Chen, Solution by Benjamin Li and Eric Shi Chen

22. **Problem:** How many factors of 2026^{2026} have exactly 2026 factors?

    Solution:

    The only prime factors of 2026 are 2 and 1013, so the prime factorization of 2026^{2026} is $2^{2026} \cdot 1013^{2026}$.

    An integer $n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots p_1ke_k$ has exactly $(e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$ factors, so in order for an integer to have 2026 factors, it must be in the form p^{2025} or $p^{1012} \cdot q$, where p and q are distinct prime numbers.

    Since the only numbers that divide 2026^{2026} are powers of 2 and 1013, the only factors that fit the form p^{2025} are 2^{2025} and 1013^{2025}, while the only factors that fit the form $p^{1012} \cdot q$ are $2^{1012} \cdot 1013$ and $2 \cdot 1013^{1012}$, for a total of $\boxed{4}$ factors of 2026^{2026}.

    Proposed by Eric Huang, Solution by Eric Huang and Eric Shi Chen

23. **Problem:** Compute $\left\lfloor (\sqrt{11} + 2\sqrt{3})^4 \right\rfloor$.

Solution: We will evaluate the expression $(\sqrt{12}+\sqrt{11})^4+(\sqrt{12}-\sqrt{11})^4$ using the binomial theorem:

$$(\sqrt{12}+\sqrt{11})^4 = \sqrt{12}^4 + \binom{4}{1}\sqrt{12}^3\cdot\sqrt{11} + \binom{4}{2}\sqrt{12}^2\cdot\sqrt{11}^2 + \binom{4}{3}\sqrt{12}\cdot\sqrt{11}^3 + \sqrt{11}^4$$
$$(\sqrt{12}-\sqrt{11})^4 = \sqrt{12}^4 - \binom{4}{1}\sqrt{12}^3\cdot\sqrt{11} + \binom{4}{2}\sqrt{12}^2\cdot\sqrt{11}^2 - \binom{4}{3}\sqrt{12}\cdot\sqrt{11}^3 + \sqrt{11}^4$$
$$\begin{aligned}(\sqrt{12}+\sqrt{11})^4+(\sqrt{12}-\sqrt{11})^4 &= 2\left[\binom{4}{0}\sqrt{12}^4 + \binom{4}{2}\sqrt{12}^2\cdot\sqrt{11}^2 + \binom{4}{4}\sqrt{11}^4\right] \\ &= 2\,(144+6\cdot 12\cdot 11+121) \\ &= 2114.\end{aligned}$$

Because $3<\sqrt{11}<\sqrt{12}<4$, we know that $0<\sqrt{12}-\sqrt{11}<1$, so $0<(\sqrt{12}-\sqrt{11})^4<1$. Therefore, we know that $(\sqrt{11}+2\sqrt{3})^4 = (\sqrt{12}+\sqrt{11})^4 = 2114-(\sqrt{12}-\sqrt{11})^4$, and finally we obtain $\left\lfloor(\sqrt{11}+2\sqrt{3})^4\right\rfloor = \boxed{2113}$.

Remark: You might be wondering what the motivation behind adding $(\sqrt{12}-\sqrt{11})^4$ is.

A way to approach this problem is to first expand the expression by the binomial theorem:

$$(\sqrt{12}+\sqrt{11})^4 = \sqrt{12}^4 + \color{red}{\binom{4}{1}\sqrt{12}^3\cdot\sqrt{11}} + \binom{4}{2}\sqrt{12}^2\cdot\sqrt{11}^2 + \color{red}{\binom{4}{3}\sqrt{12}\cdot\sqrt{11}^3} + \sqrt{11}^4$$

Notice that the terms in black are integers, while the terms in red are irrational. Terms in irrational are very annoying because we are trying to find the greatest positive integer less than the expression, and it is very time consuming to estimate the radicals in decimal.

To get rid of these red (irrational) terms, we would like to add an expression to cancel them out. This way, what we are left with is an integer. Ideally, the expression we add to cancel the red terms should be irrational, but close to 0 so we do not have to worry about more rounding issues with the floor function. We see that $(\sqrt{12}-\sqrt{11})^4$ is a perfect fit, because the subtraction sign causes the second and fourth terms to be negative:

$$(\sqrt{12}-\sqrt{11})^4 = \sqrt{12}^4 - \color{red}{\binom{4}{1}\sqrt{12}^3\cdot\sqrt{11}} + \binom{4}{2}\sqrt{12}^2\cdot\sqrt{11}^2 - \color{red}{\binom{4}{3}\sqrt{12}\cdot\sqrt{11}^3} + \sqrt{11}^4$$

Proposed by Eric Xiang, Solution by Eric Shi Chen

24. **Problem:** How many ordered pairs (a,b) of positive integers with $4\le a\le b\le 2026$ exist such that when a apples and b bananas are arranged in a row, the probability that both ends of the row have the same fruit is $\frac{1}{2}$?

Solution: Note that the probability of selecting the same fruit at both ends is entirely dependent on a and b. In particular, note that the probability that both ends of the row have the same fruit is the sum of the probabilities of both ends being apples and both ends being bananas, which is

$$\frac{a}{a+b}\cdot\frac{a-1}{a+b-1}+\frac{b}{a+b}\cdot\frac{b-1}{a+b-1}=\frac{a(a-1)+b(b-1)}{(a+b)(a+b-1)}=\frac{1}{2},$$

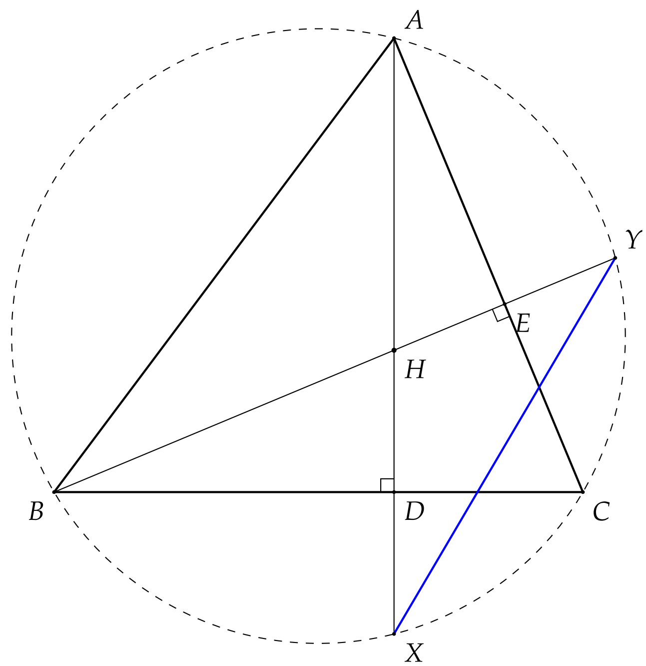
which can be simplified to $(b-a)^2=a+b$. To proceed, we define $k:=b-a$; note that since $b\ge a$, we have that $k\ge 0$. Solving the system of equations $a+b=k^2$ and $b-a=k$, we obtain that $a=\frac{k(k-1)}{2}$ and $b=\frac{k(k+1)}{2}$.

We can find the final solutions by noting that $\frac{k(k-1)}{2}\ge 4$ and $\frac{k(k+1)}{2}\le 2026$, thus $k^2-k\ge 8$ and $k^2+k\le 4052$. Finally, $4\le k\le 63$, hence our answer is $\boxed{60}$.

Proposed by Eric Huang, Solution by Aarush Kulkarni

25. **Problem:** In $\triangle ABC$, $AB = 15$, $BC = 14$, and $AC = 13$. Suppose the orthocenter of $\triangle ABC$ is H. The lines AH and BH intersect the circumcircle of $\triangle ABC$ at X and Y respectively, such that $X \neq A$ and $Y \neq B$. Given that $XY = \frac{a}{b}$ for relatively prime positive integers a and b, find $a + b$.

Solution: Observe the following diagram:



We define D as the foot of the altitude from A to BC and it is clear that X lies on the ray AD. By Heron's formula, $[ABC] = \sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84$. We also have $[ABC] = \frac{1}{2}BC \cdot AD$, and since we are given $BC = 14$, this implies that $AD = 12$.

We proceed to compute the lengths of the segments BD and CD. By the Pythagorean theorem, $CD = \sqrt{13^2 - 12^2} = 5$, so $BD = 14 - 5 = 9$. By Power of a Point on the circumcircle of $\triangle ABC$, we find that

$$AD \cdot XD = BD \cdot CD$$

Substituting our known values, we obtain $12XD = 45$, so $XD = \frac{15}{4}$.

It is well-known that the reflection of the orthocenter H across the side BC lies on the circumcircle, which implies that D is the midpoint of HX, so $HD = \frac{15}{4}$, which means $HX = \frac{15}{2}$. By the Pythagorean theorem on $\triangle BHD$, we find that

$$BH = \sqrt{\left(\frac{15}{4}\right)^2 + 9^2} = \frac{39}{4}$$

Since angles $\angle HYX$ and $\angle HAB$ subtend the same arc BX, and angles $\angle YHX$ and $\angle AHB$ are equal, this means that $\triangle YHX \sim \triangle AHB$. By virtue of this similarity, we find that

$$\frac{XY}{15} = \frac{15/2}{39/4}$$

This simplifies to $XY = 15\left(\frac{30}{39}\right) = \frac{150}{13}$, so our answer is $a + b = \boxed{163}$.

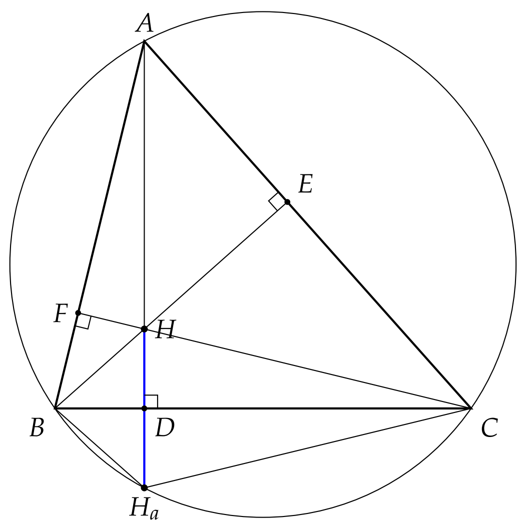
Solution 2: We label D as the foot of the altitude of A to BC and E as the foot of the altitude of B to AC. Because $\angle BDA = \angle BEA = 90^\circ$, $ABDE$ must be cyclic. Then, we see that $\angle CDE = \angle CAB$, and $\angle CED = \angle CBA$, thus $\triangle CDE \sim \triangle CAB$.

From solution 1, we know that $CD = 5$, so by equal ratios between similar triangles we obtain

$$\frac{CD}{CA} = \frac{DE}{AB} \implies \frac{5}{13} = \frac{DE}{15} \implies DE = \frac{75}{13}.$$

Since we know that the reflection of H over BC is X and the reflection of H over AC is Y, we have $\triangle HDE \sim \triangle HXY$ with a ratio of 1 : 2, so $XY = 2 \cdot DE = \frac{150}{13}$, so our final answer is $150 + 13 = \boxed{163}$.

Remark: You might be wondering why the reflection of a triangle's orthocenter over any side of the triangle must be located on its circumcircle. We can prove it as follows:



Suppose we have a triangle ABC with an orthocenter H. Let D be the foot of the altitude from A to BC, and let H_a be the reflection of H over BC. We know that $\triangle BHC \cong \triangle BH_aC$ because reflections preserve side lengths, so we have

$$\angle BHC = \angle BH_aC.$$

In quadrilateral $AFHE$, $\angle AFH$ and $\angle AEH$ are right, so we also have

$$\angle BHC = \angle FHE = 360^\circ - 90^\circ - 90^\circ - \angle BAC = 180^\circ - \angle BAC.$$

Thus,

$$\angle BH_aC + \angle BAC = 180^\circ.$$

Since the opposite angles $\angle BAC$ and $\angle BH_aC$ of quadrilateral ABH_aC sum to 180°, ABH_aC must be cyclic, which means H_a lies on the circumcircle of $\triangle ABC$.

Proposed by Eric Shi Chen, Solution by Benjamin Li, Aarush Kulkarni, and Eric Shi Chen

Accuracy Round

1. **Problem: [6]** Evaluate $(2+0)^{(2+6)}$.

   Solution: $(2+0)^{(2+6)} = 2^8 = \boxed{256}$.

   Proposed by Jonathan Ren, Solution by Jonathan Ren

2. **Problem: [7]** Amy is spinning a pencil in each hand. The pencil in her left hand spins at a rate of 4 revolutions per second, while the pencil in her right hand spins at 6 revolutions per second. How many more revolutions does the right-hand pencil make in one minute than the left-hand pencil?

   Solution: In her right hand, Amy spins her pencil $\frac{6 \text{ revolutions}}{\text{second}} \cdot 60 \text{ seconds} = 360 \text{ revolutions}$.

   In her left hand, Amy spins her pencil $\frac{4 \text{ revolutions}}{\text{second}} \cdot 60 \text{ seconds} = 240 \text{ revolutions}$

   In one minute, her right hand makes $360 \text{ revolutions} - 240 \text{ revolutions} = \boxed{120}$ revolutions more than her left hand.

   Proposed by Eric Chen, Solution by Justin Xu

3. **Problem: [8]** How many positive integers less than or equal to 2026 have an equal number of odd and even factors?

   Solution: In order for n have an equal number of odd and even factors, the power of 2 in the prime factorization of n must be exactly 1. The reasoning is as follows: take some odd factor a of n; then, we can create a one-to-one correspondence between every odd factor and every even factor by pairing a with $2a$. However, if the power of 2 is greater than 1, we get more even factors than odd factors (e.g. $a, 2a, 4a$).

   Thus, we are looking for the numbers less than or equal to 2026 with a power of 2 of exactly 1. This corresponds to the multiples of 2 that are not multiples of 4, which occurs exactly once every 4 numbers. Since 2024 is a multiple of 4 and 2026 also works, we get the total count as $\frac{2024}{4} + 1 = 506 + 1 = \boxed{507}$.

   Proposed by Jonathan Ren, Solution by Anirudh Pulugurtha

4. **Problem: [9]** Andy has a deck containing 16 cards: 4 red, 4 yellow, 4 green, and 4 blue. Every "turn", he selects 4 random cards from the deck. If they are the same color, he discards them and continues; otherwise, he returns them to the deck and shuffles it. How many expected turns will he take until he discards all cards in the deck?

   Solution: Start by selecting two cards. After picking the first card, 3 cards in the deck of 15 cards match the first card's color. Thus, the second card has a $\frac{3}{15} = \frac{1}{5}$ chance of being the same color as the first. By the same logic, the third card has a $\frac{2}{14} = \frac{1}{7}$ chance and the fourth has a $\frac{1}{13}$ chance of matching colors. Therefore, the chance of first discarding one set of same-colored cards in the deck is $\frac{1}{5} \cdot \frac{1}{7} \cdot \frac{1}{13} = \frac{1}{455}$, implying Andy expects to take 455 turns to discard his first four cards.

   A similar process can be used on each other color set, only with the number of cards left in the deck decreasing for each same-colored set completed. Each turn taken now, with 12 cards in the deck, has a $\frac{3}{11} \cdot \frac{2}{10} \cdot \frac{1}{9} = \frac{1}{11 \cdot 5 \cdot 3} = \frac{1}{165}$ chance of completing a set.

   For the third row, Andy has a $\frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{35}$ chance of completing the set.

   Finally, one final turn discards the only remaining set.

   Therefore, Andy is expected to take $455 + 165 + 35 + 1 = \boxed{656}$ turns to discard all 16 cards.

   Proposed by Jonathan Ren, Solution by Jonathan Ren

5. **Problem:** **[10]** The functions $f(x) = \sqrt{x^3 - 7x^2 + 17x - 14}$ and $g(x) = \sqrt{3x - 6}$ are graphed in the real xy-plane. Find the sum of the x coordinates of all the points where the graphs of $f(x)$ and $g(x)$ intersect.

   Solution: In order to find the points of intersection, we set $f(x) = g(x)$ and solve:

$$\sqrt{x^3 - 7x^2 + 17x - 14} = \sqrt{3x - 6}$$
$$x^3 - 7x^2 + 17x - 14 = 3x - 6$$
$$x^3 - 7x^2 + 14x - 8 = 0$$
$$(x-1)(x-2)(x-4) = 0$$

   We see that $x = 1, 2, 4$ are solutions. However, since we squared both sides of the equation, we have to make sure that none of the expressions inside the square roots are negative. So, we have to plug these three values of x into the functions to verify that they are in the domains of the functions.

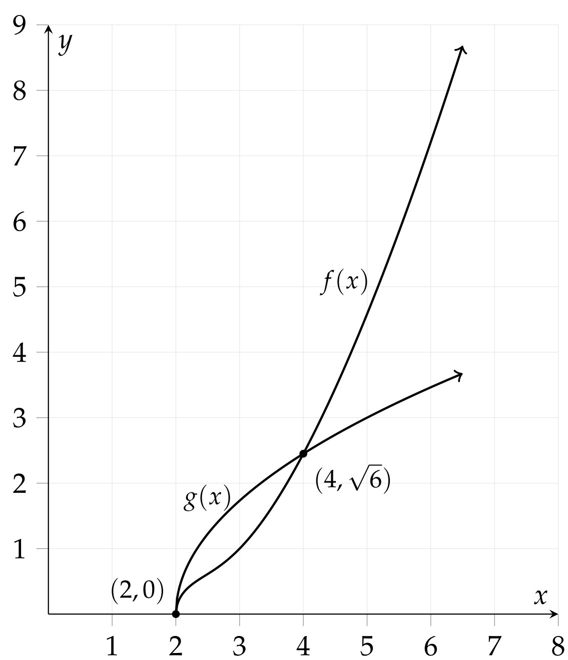
   When $x = 1$, $f(1) = \sqrt{1^3 - 7 \cdot 1^2 + 17 \cdot 1 - 14} = \sqrt{-3}$ and $g(1) = \sqrt{3 \cdot 1 - 6} = \sqrt{-3}$. Since these functions are graphed in the real plane, $x = 1$ is not in the domain of either function and is an extraneous solution.

   When $x = 2$, $f(2) = \sqrt{2^3 - 7 \cdot 2^2 + 17 \cdot 2 - 14} = 0$ and $g(2) = \sqrt{3 \cdot 2 - 6} = 0$. So, when $x = 2$, $f(x)$ and $g(x)$ intersect at the point $(2, 0)$.

   When $x = 4$, $f(2) = \sqrt{4^3 - 7 \cdot 4^2 + 17 \cdot 4 - 14} = \sqrt{6}$ and $g(2) = \sqrt{3 \cdot 4 - 6} = \sqrt{6}$. So, when $x = 4$, $f(x)$ and $g(x)$ intersect at the point $(4, \sqrt{6})$.

   Our final answer is $2 + 4 = \boxed{6}$.

   Remark: Here is a graph showing the two intersection points of $f(x)$ and $g(x)$.

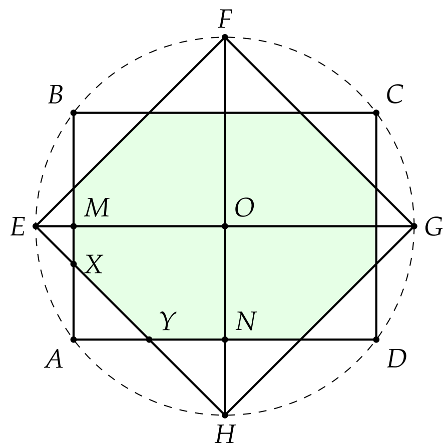


   Proposed by Eric Shi Chen, Solution by Eric Shi Chen

6. **Problem:** **[11]** Rectangle $ABCD$ is inscribed in a circle. Points E, F, G, and H are the midpoints of minor arcs AB, BC, CD, and DA. Given that $AB = 6$ and $BC = 8$, what is the area of the overlap between $ABCD$ and $EFGH$?

   Note: In this problem, the overlap is defined as the intersection of $ABCD$ and $EFGH$, not the union.

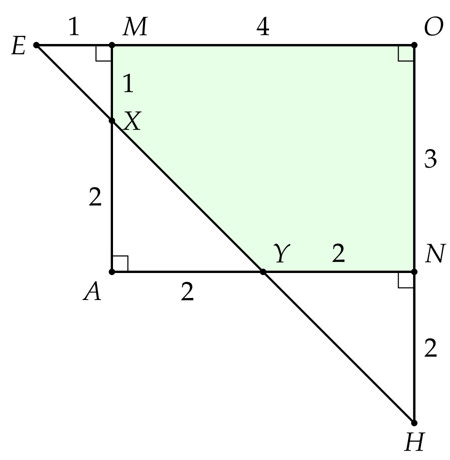
   Solution:



Let O be the circumcenter of rectangle $ABCD$, and let R be the circumradius. We know that O is the point of intersection of diagonals $\overline{AC}$ and $\overline{AC}$, and O is the midpoint of both segments. Since $AC = \sqrt{AB^2 + BC^2} = \sqrt{6^2 + 8^2} = 10$, we know that $R = AO = CO = \frac{10}{2} = 5$.

Now, consider point E. Since E is the midpoint of the minor arc AB, E is on the perpendicular bisector of $\overline{AB}$. Similarly, G is on the perpendicular bisector of $\overline{CD}$. We know that the segment that connects the midpoints of opposite sides passes through the center of the rectangle and is a perpendicular bisector of the opposite sides, so $\overline{EG}$ passes through O and is the perpendicular bisector of $\overline{AB}$ and $\overline{CD}$. Similarly, we know that $\overline{FH}$ passes through O and is the perpendicular bisector of $\overline{AB}$ and $\overline{CD}$. Furthermore, since $\overline{AB} \perp \overline{BC}$, we know that $\overline{EG} \perp \overline{FH}$, so $EFGH$ is a square.

Next, let M and N be the midpoints of $\overline{AB}$ and $\overline{AD}$, respectively, and let X and Y be the intersection between $\overline{EH}$ and segments $\overline{AB}$ and $\overline{AD}$, respectively. In order to find the area of overlap between $ABCD$ and $EFGH$, we need to calculate the octagon-shaped area shaded in green. By symmetry, we know that the area of pentagon $MXYNO$ is a quarter of the area of the green shaded region.



We know that EOH is an isosceles right triangle with legs of length 5, and by parallel lines we know that EMX and YNH are also isosceles right triangles. We know that $MO = \frac{BC}{2} = 4$, and $NO = \frac{AB}{2} = 3$, so $MX = EM = EO - MO = 1$ and $YN = HN = HO - NO = 2$. Therefore, $AX = AM - MX = 2$, and $AY = AN - YN = 2$. We can then calculate $[MXYNO] = [AMON] - [AXY] = 4 \cdot 3 - \frac{2 \cdot 2}{2} = 10$.

So, the area of overlap between $ABCD$ and $EFGH$ is $4[MXYNO] = \boxed{40}$.

Proposed by Tanish Parida, Solution by Eric Shi Chen

7. **Problem: [11]** Begin with a line segment in the coordinate plane whose endpoints are $(0,0)$ and $(5,0)$. Rotate the segment counterclockwise about $(5,0)$ until the other endpoint lies on a lattice point. That lattice point becomes the new pivot, and the segment is again rotated counterclockwise about this pivot until the opposite endpoint reaches another lattice point. This process continues indefinitely.

   Let the initial pivot $(5,0)$ be the first pivot point. The 2026^{th} pivot point is (m,n), where m and n are integers. Compute $|m+n|$.

   Solution: Suppose the first line segment is labeled with endpoints a_0 and a_1 with $a_0 = (0,0)$ and $a_1 = (5,0)$.

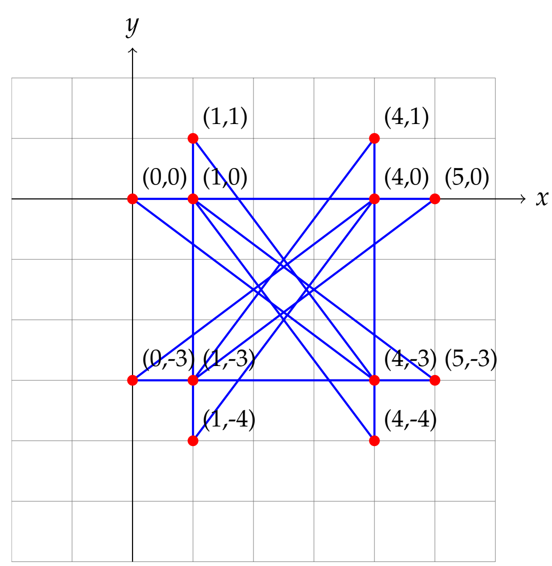
   Rotating $(0,0)$ clockwise about $(5,0)$, we find that the first lattice point reached is $(1,-3)$. So, we let the next pivot point be $a_2 = (1,-3)$.

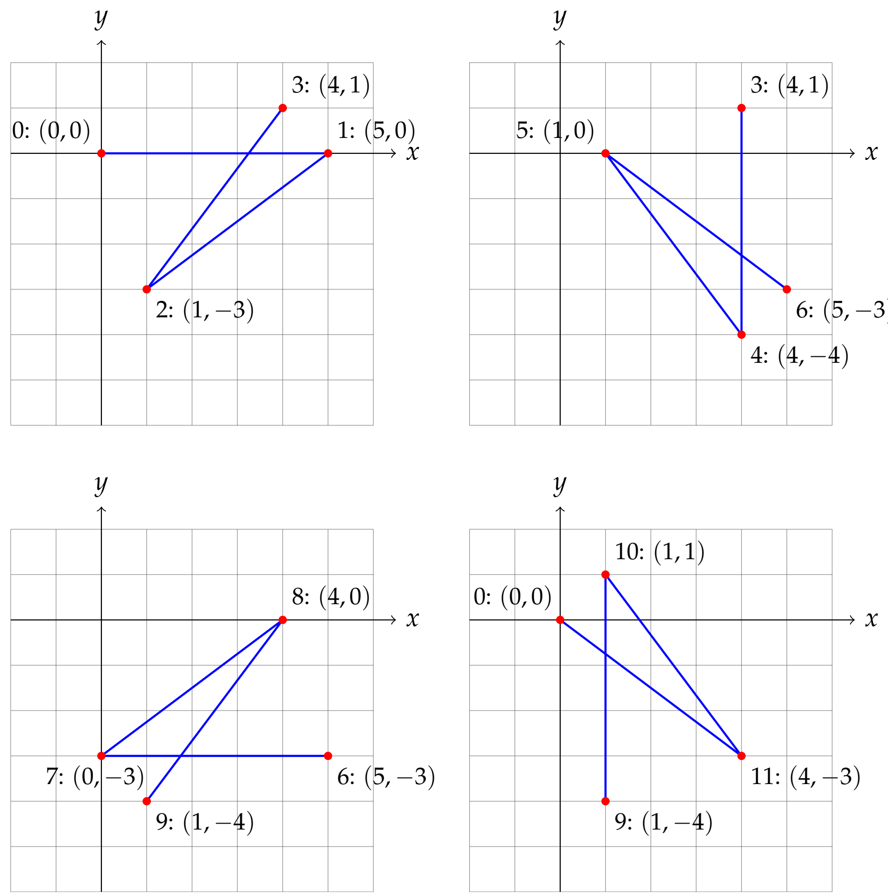
   We can repeat this process to find:

$$\begin{aligned} a_3 &= (4,1) \\ a_4 &= (4,-4) \\ a_5 &= (1,0) \\ a_6 &= (5,-3) \\ a_7 &= (0,-3) \\ a_8 &= (4,0) \\ a_9 &= (1,-4) \\ a_{10} &= (1,1) \\ a_{11} &= (4,-3) \\ a_{12} &= (0,0) \\ a_{13} &= (5,0). \end{aligned}$$

   Notice that the pivot points repeat, specifically $a_i = a_j$ if $i \equiv j \pmod{12}$. Since $2026 \equiv 10 \pmod{12}$, and $a_{10} = (1,1)$, we have $a_{2026} = (1,1)$ so $|m+n| = \boxed{2}$.

   Remark: A visualization of the process is shown below.





Proposed by Nathan Tan, Solution by Ben Li and Eric Shi Chen

8. **Problem: [12]** Let a and b be factors of 11!. The probability that a is divisible by b is equal to $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $10m + n$.

   Solution: First, expand 11! and prime factorize it to get $11! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$.

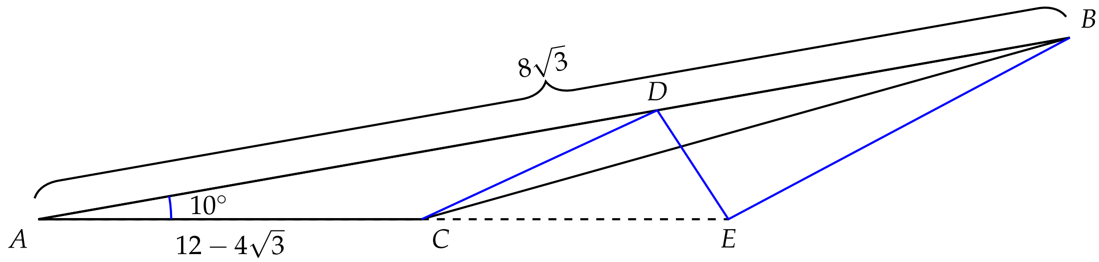
   Let $\nu_p(x)$ denote the highest power of p that divides x. So, in order for $b \mid a$, then $\nu_2(a) \geq \nu_2(b)$, $\nu_3(a) \geq \nu_3(b)$, $\nu_5(a) \geq \nu_5(b)$, $\nu_7(a) \geq \nu_7(b)$, and $\nu_{11}(a) \geq \nu_{11}(b)$. The probabilities of each of these inequalities are independent because they involve distinct prime numbers.

   For each of those functions, all possible values have equal probabilities of occurring, meaning the inequalities respective probabilities of occurring are $\frac{5}{9}$, $\frac{3}{5}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{3}{4}$ respectively. So, the total probability that $b \mid a$ is $\frac{5 \cdot 3 \cdot 2 \cdot 3 \cdot 3}{9 \cdot 5 \cdot 3 \cdot 4 \cdot 4} = \frac{1}{8}$, so $10m + n = \boxed{18}$.

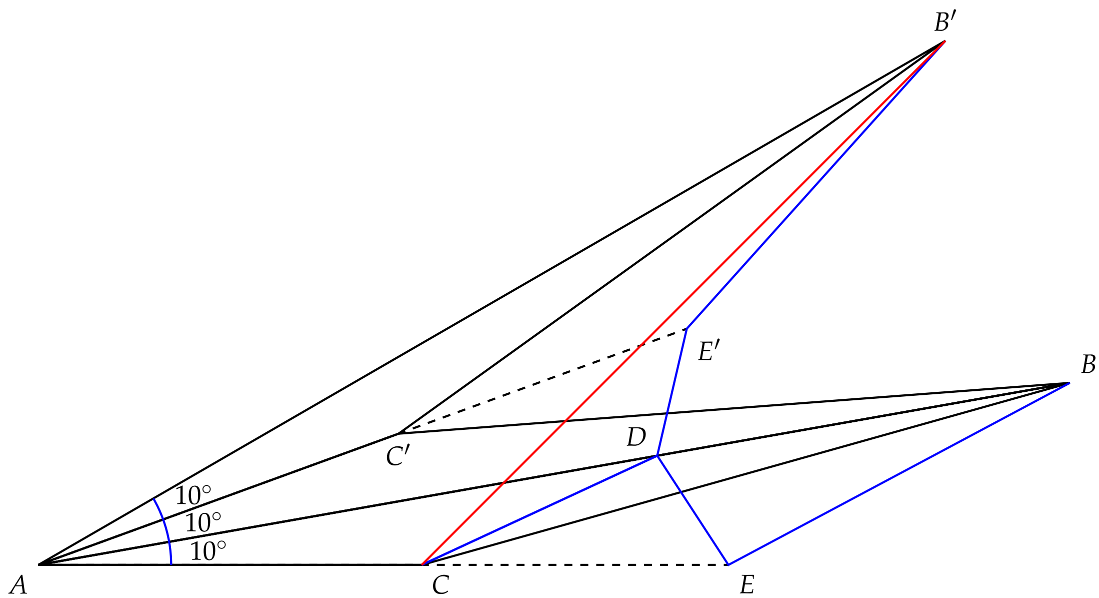
   Proposed by Eric Huang, Solution by Eric Huang

9. **Problem: [13]** In triangle ABC, $AB = 8\sqrt{3}$, $AC = 12 - 4\sqrt{3}$, and $\angle BAC = 10^\circ$. Suppose point D is on line AB and point E is on line AC. Let m be the minimum possible value of $CD + DE + EB$. Compute m^2.

   Note: D is not necessarily between A and B, and E is not necessarily between A and C. The points D and E in the diagram below are arbitrarily selected.



Solution:



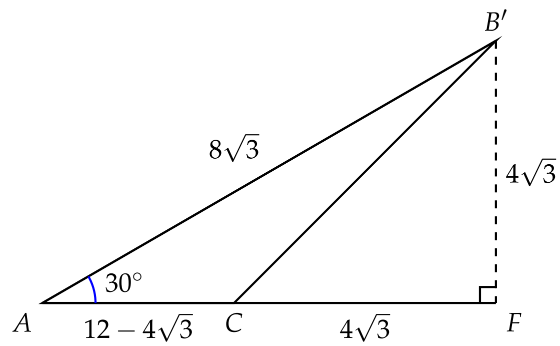
For any choice of points D and E on AB and AC, respectively, we can minimize the value of $CD + DE + EB$ by using reflections. Note that when we reflect a segment, its length is preserved.

First, we reflect C and E across AB to C' and E', respectively. By the triangle inequality, $CD + DE = CD + DE' \geq CE'$, so in order to minimize $CD + DE$ we must have D be the intersection of AB and CE'. Next, we reflect B across AC' to B'. By the triangle inequality, $DE + EB = DE' + E'B \geq DB$, so in order to minimize $DE + EB$ we must have E' be the intersection of $B'D$ and AC'.

We notice that in order to minimize $CD + DE + EB$, we must have point D be on CE' and point E' be on $B'D$, and the only way for this to be true is for C, D, E', and B' to be collinear. Intuitively, we can see in the diagram above that $CD + DE' + E'B'$ is minimized if it is "smoothed out" to be CB', as shown in red.

Therefore, the minimum value m of $CD + DE + EB$ is simply the length of CB'.

Notice that since we reflected twice, $\angle B'AC = 30°$.



We can now finish the problem in two ways:

First, we drop the perpendicular from B' to meet line AC at point F. Then, $AB'F$ is a 30-60-90 right triangle, so the hypotenuse is $AB' = AB = 8\sqrt{3}$. Then we have $B'F = 4\sqrt{3}$ and $AF = 12$. Using this in combination with $AC = 12 - 4\sqrt{3}$ gives $CF = 4\sqrt{3}$. Finally, using the Pythagorean Theorem on $\triangle CB'F$ gives $m^2 = (CB')^2 = (4\sqrt{3})^2 + (4\sqrt{3})^2 = \boxed{96}$.

Alternatively, we can calculate $(CB')^2$ directly using the law of cosines:

$$\begin{aligned} m^2 = (CB')^2 &= (AB')^2 + (AC)^2 - 2 \cdot AB \cdot AC \cdot \cos \angle B'AC \\ &= (8\sqrt{3})^2 + (12 - 4\sqrt{3})^2 - 2(8\sqrt{3})(12 - 4\sqrt{3}) \cos 30^\circ \\ &= 192 + (144 + 48 - 96\sqrt{3}) - 2(8\sqrt{3})(12 - 4\sqrt{3}) \cdot \tfrac{\sqrt{3}}{2} \\ &= 192 + (192 - 96\sqrt{3}) - (8\sqrt{3})(12 - 4\sqrt{3})\sqrt{3} \\ &= 384 - 96\sqrt{3} - 8(12 - 4\sqrt{3}) \cdot 3 \\ &= 384 - 96\sqrt{3} - 288 + 96\sqrt{3} \\ &= \boxed{96}. \end{aligned}$$

Proposed by Eric Shi Chen, Solution by Nathan Tan and Eric Shi Chen

10. **Problem: [13]** Ben and Brandon are playing a game where they repeatedly flip a coin. Ben wins if 1 heads followed by 2024 tails appears. Brandon wins if 2025 heads come in a row. They will flip a coin until someone wins.

    If the probability that Ben wins is $\dfrac{2^a - 1}{b \cdot 2^c - 1}$, find $a + b + c$.

    Solution: We define p_k to be the probability that Ben wins given the last k flips are heads, and q_k the probability that Ben wins given the last flips are a head followed by k tails, for $k \in \{1, \ldots, 2024\}$. The game effectively begins when the first head is flipped; thus, the overall probability Ben wins is exactly p_1.

    Note the following:

    - From a state of k consecutive heads, the next flip is either a head or a tail; in particular, flipping a tail transitions the game to the state for q_1, thus

    $$p_k = \frac{1}{2} p_{k+1} + \frac{1}{2} q_1$$

      for $k \in \{1, \ldots, 2024\}$, or hence $p_{2025} = 0$. Rewriting the recurrence for q_k as $q_{k+1} - p_1 = 2(q_k - p_1)$ implies the sequence is geometric; hence we find $q_{2024} - p_1 = 2^{2023}(q_1 - p_1)$, and since $q_{2024} = 1$, we obtain

    $$1 - p_1 = 2^{2023}(q_1 - p_1)$$

- Similarly, from a state of a head followed by k tails, flipping a head transitions the game to the state for p_1, hence we obtain

$$q_k = \frac{1}{2}q_{k+1} + \frac{1}{2}p_1$$

for $k \in \{1, \ldots, 2023\}$, with $q_{2024} = 1$. We rewrite the recurrence for p_k as $p_{k+1} - q_1 = 2(p_k - q_1)$; this, in tandem with $p_{2025} = 0$, guarantees that

$$-q_1 = 2^{2024}(p_1 - q_1)$$

The previous equation simplifies to $q_1(2^{2024} - 1) = 2^{2024}p_1$, which implies $q_1 - p_1 = \frac{p_1}{2^{2024}-1}$. Substituting this into our equation for $1 - p_1$, we find that

$$\begin{aligned} 1 - p_1 &= 2^{2023}\left(\frac{p_1}{2^{2024}-1}\right) \\ 1 &= p_1\left(\frac{2^{2024} - 1 + 2^{2023}}{2^{2024}-1}\right) \\ 1 &= p_1\left(\frac{3 \cdot 2^{2023} - 1}{2^{2024}-1}\right). \end{aligned}$$

Thus, we obtain $p_1 = \frac{2^{2024}-1}{3 \cdot 2^{2023}-1}$, hence $a + b + c = \boxed{4050}$.

Proposed by Eric Huang, Solution by Aarush Kulkarni

11. **Problem: [Tiebreaker]** Find the positive integer closest to the following:

$$\sum_{n=1}^{2026}\left(n^{0.2026}\right)$$

(This question serves as a **tiebreaker** for the individual rounds.)

Solution: We approximate the sum as an *integral*, a calculus concept that denotes the area under the curve. To simplify computations, we find the integral from 1 to 2026 of $n^{0.2}$, denoted as

$$\int_{1}^{2026} n^{0.2}\, dn.$$

Next, we apply the *power rule*, which states that

$$\int x^n\, dx = \frac{x^{n+1}}{n+1}$$

for $n \neq -1$. Thus, the integral evaluates to

$$\frac{n^{1.2}}{1.2}.$$

To compute a *definite integral*, we subtract the value of the above function, known as the *antiderative*, for the limits. We get our approximation as

$$\frac{2026^{1.2} - 1^{1.2}}{1.2}.$$

After computing, we obtain 7740, which is quite close to the actual answer of 7880.

Proposed by Anirudh Pulugurtha, Solution by Anirudh Pulugurtha

Team Round

Round 1

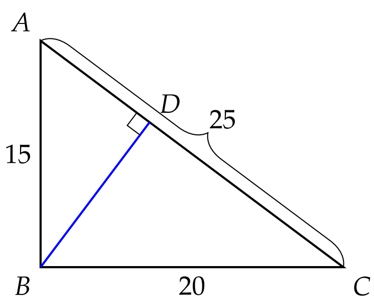
1. **Problem: [9]** We can define the operator ♣ that takes in integers a and b as: $a♣b = a^2 - b^2$. What is the value of the expression $(2♣1) + (4♣3)$?

   Solution: We plug these values into the formula defined. Note that order of operations means we evaluate the ♣ operations first.

   $$(2♣1) + (4♣3) = (2^2 - 1^2) + (4^2 - 3^2) = (4 - 1) + (16 - 9) = 3 + 7 = \boxed{10}.$$

   Proposed by Aarush Kulkarni, Solution by Anirudh Pulugurtha

2. **Problem: [9]** In triangle ABC, $AB = 15$, $BC = 20$, and $CA = 25$. D is the foot of the altitude from B to AC. What is the length of BD?



   Solution: We can calculate the area of triangle $\triangle ABC$ in two different ways.

   First, we know that since ABC is a right triangle, its area is the products of the lengths of the legs divided by 2. We find that the area of $\triangle ABC$ is $[ABC] = \frac{15 \cdot 20}{2} = 150$.

   Next, the area of a triangle can also be found by multiply a base by the corresponding height and dividing by two. Treating AC as the base and BD as the unknown height we are solving for, we find that the area of $\triangle ABC$ is $[ABC] = \frac{25 \cdot BD}{2}$.

   Setting these two equations as equal, we obtain

   $$[ABC] = \frac{15 \cdot 20}{2} = \frac{25 \cdot BD}{2} = 150 \implies BD = \boxed{12}.$$

   Proposed by Eric Shi Chen, Solution by Tanish Parida and Eric Shi Chen

3. **Problem: [9]** On Week 1, Richard has 20 coins and Nathan has 10 coins. Each subsequent week, Richard finds 100 coins while Nathan somehow doubles the number of coins he has. If n is the first week in which Nathan's total exceeds Richard's, find the value of n.

   Solution: We can figure this out by writing out the number of coins they have each week:

   - In week 1, we are given that Richard has 20 coins while Nathan has 10.
   - In week 2, Richard has 120 and Nathan has 20.
   - In week 3, Richard has 220 and Nathan has 40.
   - In week 4, Richard has 320 and Nathan has 80.
   - In week 5, Richard has 420 and Nathan has 160.
   - In week 6, Richard has 520 and Nathan has 320.
   - In week 7, Richard has 620 and Nathan has 640.

Since $640 > 620$, the first week where Nathan's total exceeds Richards is week $\boxed{7}$.

Proposed by Eric Shi Chen, Solution by Anirudh Pulugurtha

Round 2

1. **Problem: [10]** What is the area of a circle that has a circumference of $12\sqrt{\pi}$?

   Solution: We can use the circle's circumference to solve for its radius:

$$C = 2\pi r = 12\sqrt{\pi}$$
$$r = \frac{6\sqrt{\pi}}{\pi} = \frac{6}{\sqrt{\pi}}.$$

   Using this, we find the area:

$$A = \pi r^2 = \pi\left(\frac{6}{\sqrt{\pi}}\right)^2 = \pi\left(\frac{36}{\pi}\right) = \boxed{36}.$$

   Proposed by Eric Shi Chen, Solution by Jonathan Ren

2. **Problem: [10]** How many integers less than 2026 are squares of primes?

   Solution: First we note that the smallest perfect square less than 2026 is $2025 = 45^2$. Thus, the problem is reduced to counting the number of primes less than or equal to 45. We can just list them out:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.$$

   Counting, we get that there are 14 primes less than or equal to 14, and thus $\boxed{14}$ squares of primes less than 2026.

   Proposed by Ayaan Garg, Solution by Anirudh Pulugurtha

3. **Problem: [10]** Jonathan rolls three fair six-sided dice and adds their results. Let the probability that their sum is even be $\frac{a}{b}$ such that a and b are relatively prime integers. Find $a + b$.

   Solution: We can start by looking at 2 dice. For any number we roll on the first die, we have a $\frac{1}{2}$ chance of making the sum either even or odd with the second die. Adding in the third die, we can see that the sum would still be even or odd with $\frac{1}{2}$ probability. This means our answer is $1 + 2 = \boxed{3}$.

   Proposed by Eric Shi Chen, Solution by Tanish Parida

Round 3

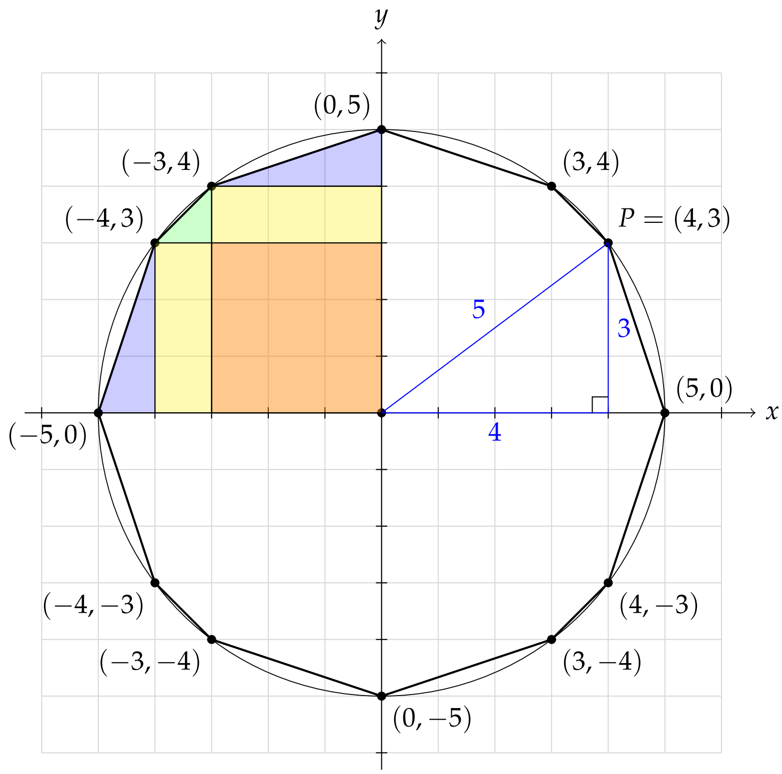
1. **Problem: [12]** Benjamin is designing a table for his new club, which he is calling "Knights of the Dodecagonal Table". His table design is made by taking all points with integer coordinates on the graph $x^2 + y^2 = 25$ and connecting them to form a concave polygon. What is the area of this new table, in square units?

   Solution: The graph $x^2 + y^2 = 25$ is a circle centered at the origin with a radius of $\sqrt{25} = 5$. Clearly, the points $(5, 0), (0, 5), (-5, 0), (0, -5)$ are all on this graph.

   Now, consider a point P on this circle. If we drop an altitude from point P to the x-axis, a triangle with hypotenuse 5 is formed, and the lengths of the legs are equal to the absolute value of the coordinates of P. Therefore, P only has integer coordinates when the lengths of the legs are integers. The only Pythagorean triple which satisfies such a right triangle is the $3-4-5$ triple. Thus, the other 8 points are $(4, 3), (3, 4), (-3, 4), (-4, 3), (-4, -3), (-3, -4), (3, -4)$, and $(4, -3)$.

   By dividing the resulting dodecagon into triangles and rectangles, then summing their areas, we find the area of the whole table to be $\boxed{74}$.

   Below is Benjamin's table graphed on the xy-plane. $P = (4, 3)$ is used as an example, and we can find the area of a quarter of the table to be $3 \cdot 3 + 3 \cdot 1 + 3 \cdot 1 + \frac{3 \cdot 1}{2} + \frac{3 \cdot 1}{2} + \frac{1 \cdot 1}{2} = \frac{37}{2}$, as show in the second quadrant.



Proposed by Jonathan Ren, Solution by Jonathan Ren and Eric Shi Chen

2. **Problem: [12]** Nathan and Eric and 43 other people were arguing about how they should slice a cake. Given that 9 people want x slices of cake, 8 people want x^2 slices of cake, 7 people want x^3 slices of cake and so on until 1 person wants x^9 slices of cake, find x if the total number of slices is 2026.

   Solution: Due to how quickly the x^9 term scales, it suffices to quickly check small values of x.

   When $x = 1$, there are only 45 slices of cake, which is far too few. When $x = 3$, the x^9 term becomes $3^9 = 27^3$. This is significantly larger than 20^3, and $20^3 = 8000 > 2026$, so x cannot equal a number greater than 3. Therefore, the only possible value for x is $\boxed{2}$.

   Proposed by Tanish Parida, Solution by Jonathan Ren

3. **Problem: [12]** Given $2^{33} = 10^k$, find k, rounded to the nearest integer.

   Solution: Conveniently, $\log_{10}(2) = 0.30103\cdots \approx \frac{3}{10}$. We take the base 10 logarithm of both sides, then apply the power rule of logarithms:

$$\begin{aligned} \log_{10}(2^{33}) &= \log_{10}(10^k) \\ 33\log_{10}(2) &= k \\ \tfrac{99}{10} &\approx k, \end{aligned}$$

   and rounding up $k \approx \frac{99}{10}$ results in an answer of $\boxed{10}$.

   Alternatively, note that $2^{10} = 1024 \approx 10^3$. Therefore,

$$\begin{aligned} 2^{33} = 2^3 \cdot \left(2^{10}\right)^3 &= 10^k \\ 8 \cdot 10^9 &\approx 10^k. \end{aligned}$$

   $8 \cdot 10^9$ is clearly much closer to 10^{10} than 10^9, and we can comfortably round up since we estimated 2^{10} as smaller than it actually is. So, $10^{10} \approx 10^k$ and $k = \boxed{10}$.

Remark: The actual value of k is around 9.93399.

Proposed by Jonathan Ren, Solution by Jonathan Ren

Round 4

1. **Problem: [14]** Suppose $a = 20! \cdot 21! \cdot 22! \cdot 23! \cdot 24! \cdot 25!$. If 2^n is the largest power of two that divides a and m^{26} is the largest 26th power that divides a, find $m + n$.

   Solution:

   The prime factorization of 20! is $2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$

   The prime factorization of 21! is $2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$

   The prime factorization of 22! is $2^{19} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$

   The prime factorization of 23! is $2^{19} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

   The prime factorization of 24! is $2^{22} \cdot 3^{10} \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

   The prime factorization of 25! is $2^{22} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$

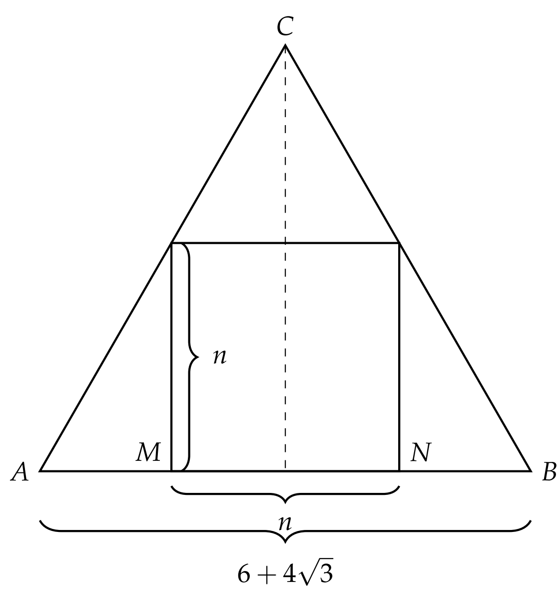
   Therefore, the prime factorization of $a = 20! \cdot 21! \cdot 22! \cdot 23! \cdot 24! \cdot 25!$ is equal to $2^{118} \cdot 3^{55} \cdot 5^{26} \cdot 7^{17} \cdot 11^{10} \cdot 13^6 \cdot 17^6 \cdot 19^6 \cdot 23^3$

   We see that the largest power of 2 that divides a is 2^{118}, which means $n = 118$. The largest 26th power that divides a is $m = 2^{\lfloor \frac{118}{26} \rfloor} \cdot 3^{\lfloor \frac{55}{26} \rfloor} \cdot 5^{\lfloor \frac{26}{26} \rfloor} = 2^4 \cdot 3^2 \cdot 5 = 720$. Therefore, $m + n = 720 + 118 = \boxed{838}$.

   Proposed by Eric Huang, Solution by Tanish Parida and Eric Li

2. **Problem: [14]** Find the area of a square inscribed in an equilateral triangle, such that one edge of the square is on the edge of the triangle if the side length of the triangle is $6 + 4\sqrt{3}$.

   Solution: Consider the following diagram:



   Note that $\triangle ABC$ is an equilateral triangle. Therefore, we can calculate the length of AB in terms of n by using $30 - 60 - 90$ triangle ratios with AM and NB, to find that $AM = NB = \frac{n}{\sqrt{3}}$. Then

$$AB = AM + MN + NB$$
$$6 + 4\sqrt{3} = \frac{n}{\sqrt{3}} + n + \frac{n}{\sqrt{3}}.$$

Solving, we get $n = 6$, and thus the area of the square is $\boxed{36}$.

Proposed by Ayaan Garg, Solution by Tanish Parida and Jonathan Ren

3. **Problem: [14]** Jonathan and Nathan are playing a card game. Each turn, the player draws one card from a standard 52-card deck plus 2 jokers. If the player draws a King or Queen, they win immediately, but if they draw a Joker, then they lose instantly. Jonathan goes first, then they alternate turns. The probability that Jonathan wins the game can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

   Solution: Let P be the probability Jonathan wins. We can group all winning outcomes for Jonathan into the following three cases.

   - **Case 1**: Jonathan draws a King or Queen on his first turn. This has a $\frac{8}{54} = \frac{4}{27}$ chance of occurring.
   - **Case 2**: Jonathan draws a card that is not a King, Queen, or Joker, and Nathan draws a Joker on his first turn. The chance of this is $\frac{44}{54} \cdot \frac{2}{54} = \frac{22}{729}$.
   - **Case 3**: Jonathan wins the game in a later round. If both Jonathan and Nathan draw no Kings, Queens, or Jokers, the scenario is the exact same as it was at the beginning of the game. Thus, the chance Jonathan wins after the first round is $(\frac{44}{54})^2 P = \frac{484}{729}P$.

   We can now add these three probabilities and solve for P to get our answer:

$$P = \frac{4}{27} + \frac{22}{729} + \frac{484}{729}P$$
$$\frac{245}{729}P = \frac{130}{729}$$
$$P = \frac{130}{245} = \frac{26}{49}.$$

   So, our final answer is $26 + 49 = \boxed{75}$.

   Proposed by Jonathan Ren, Solution by Jonathan Ren

Round 5

1. **Problem: [16]** Players 1, 2, 3, and 4 are playing a tennis bracket tournament. They are randomly paired for the first round, and the winners of each game will play each other in the finals. When Player a plays Player b, Player a has a $\frac{a^2+ab}{(a+b)^2}$ chance of winning. However, Player 4 has to play with his left hand for one game, decreasing his skill to the same level that Player 2 has. Player 4 can choose which hand to use based on who he plays first.

   If Player 4 uses the optimal strategy, the probability that Player 4 wins the tournament can be expressed as $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $p + q$.

   Solution: First, notice that $\frac{a^2+ab}{(a+b)^2}$ can be simplified to $\frac{a}{a+b}$. For the sake of clarity, we rename players 1, 2, 3, and 4 as p_1, p_2, p_3, and p_4, respectively. We will now proceed with casework. Notice that there are three cases; we will group them by whomever p_4 plays in the first round:

   - p_4 plays p_3 in the 1st round, and then p_1 or p_2 in the next.
   - p_4 plays p_2 in the 1st round, and then p_1 or p_3 in the next.
   - p_4 plays p_1 in the 1st round, and then p_2 or p_3 in the next.

   Then, for each case we will calculate the probability that p_4 wins depending on when he uses his left hand.

   Case 1: p_4 plays p_3 in the 1st round.

   Case 1a: p_4 uses his left hand in the 1st round and his right hand in the 2nd round.

In the 1st round, p_4 has a $\frac{2}{5}$ chance of beating p_3. In the 2nd round p_4 has a $\frac{4}{5}$ chance of beating p_1 and a $\frac{4}{6}$ chance of beating p_2. p_1 will beat p_2 in the 1st round (and thus show up to face p_4 in the 2nd round) with a $\frac{1}{3}$ chance, and p_2 will beat p_1 in the 1st round with a $\frac{2}{3}$ chance.

In order to calculate the chance p_4 wins in the 2nd round, we have to first consider the chances of p_1 and p_2 showing up, and then multiply that by the chance p_4 beats p_1 and the chance p_4 beats p_2, respectively, and then add the two products together.

So, the probability p_4 wins the tournament is

$$\frac{2}{5} \cdot \left(\frac{1}{3} \cdot \frac{4}{5} + \frac{2}{3} \cdot \frac{4}{6}\right) = \frac{64}{225}.$$

Case 1b: p_4 uses his right hand in the 1st round and his left hand in the 2nd round.

In the 1st round, p_4 has a $\frac{4}{7}$ chance of beating p_3. In the 2nd round p_4 has a $\frac{2}{3}$ chance of beating p_1 and a $\frac{2}{4}$ chance of beating p_2. In the 1st round, p_1 will beat p_2 with a $\frac{1}{3}$ chance, and p_2 will beat p_1 a $\frac{2}{3}$ chance.

So, the probability p_4 wins the tournament is

$$\frac{4}{7} \cdot \left(\frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{2}{4}\right) = \frac{20}{63}.$$

We see that $\frac{20}{63} > \frac{64}{225}$, so p_4 should play with his right hand first if he plays p_3 in the 1st round.

Case 2: p_4 plays p_2 in the 1st round.

Case 2a: p_4 uses his left hand in the 1st round and his right hand in the 2nd round.

In the 1st round, p_4 has a $\frac{2}{4}$ chance of beating p_2. In the 2nd round p_4 has a $\frac{4}{5}$ chance of beating p_1 and a $\frac{4}{7}$ chance of beating p_3. In the 1st round, p_1 will beat p_3 with a $\frac{1}{4}$ chance, and p_3 will beat p_1 a $\frac{3}{4}$ chance.

So, the probability p_4 wins the tournament is

$$\frac{2}{4} \cdot \left(\frac{1}{4} \cdot \frac{4}{5} + \frac{3}{4} \cdot \frac{4}{7}\right) = \frac{11}{35}.$$

Case 2b: p_4 uses his right hand in the 1st round and his left hand in the 2nd round.

In the 1st round, p_4 has a $\frac{4}{6}$ chance of beating p_2. In the 2nd round p_4 has a $\frac{2}{3}$ chance of beating p_1 and a $\frac{2}{5}$ chance of beating p_3. In the 1st round, p_1 will beat p_3 with a $\frac{1}{4}$ chance, and p_3 will beat p_1 a $\frac{3}{4}$ chance.

So, the probability p_4 wins the tournament is

$$\frac{4}{6} \cdot \left(\frac{1}{4} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{2}{5}\right) = \frac{14}{45}.$$

We see that $\frac{11}{35} > \frac{14}{45}$, so p_4 should play with his left hand first if he plays p_2 in the 1st round.

Case 3: p_4 plays p_1 in the 1st round.

Case 3a: p_4 uses his left hand in the 1st round and his right hand in the 2nd round.

In the 1st round, p_4 has a $\frac{2}{3}$ chance of beating p_1. In the 2nd round p_4 has a $\frac{4}{6}$ chance of beating p_2 and a $\frac{4}{7}$ chance of beating p_3. In the 1st round, p_2 will beat p_3 with a $\frac{2}{5}$ chance, and p_3 will beat p_2 a $\frac{3}{5}$ chance.

So, the probability p_4 wins the tournament is

$$\frac{2}{3} \cdot \left(\frac{2}{5} \cdot \frac{4}{6} + \frac{3}{5} \cdot \frac{4}{7}\right) = \frac{128}{315}.$$

Case 3b: p_4 uses his left hand in the 1st round and his right hand in the 2nd round.

In the 1st round, p_4 has a $\frac{4}{5}$ chance of beating p_1. In the 2nd round p_4 has a $\frac{2}{4}$ chance of beating p_2 and a $\frac{2}{5}$ chance of beating p_3. In the 1st round, p_2 will beat p_3 with a $\frac{2}{5}$ chance, and p_3 will beat p_2 a $\frac{3}{5}$ chance.

So, the probability p_4 wins the tournament is

$$\frac{4}{5} \cdot \left(\frac{2}{5} \cdot \frac{2}{4} + \frac{3}{5} \cdot \frac{2}{5}\right) = \frac{44}{125}.$$

We see that $\frac{128}{315} > \frac{44}{125}$, so p_4 should play with his left hand first if he plays p_1 in the 1st round.

Each of these 3 cases are equally likely to occur, so if p_4 plays optimally, he has a $\frac{1}{3}(\frac{20}{63} + \frac{11}{35} + \frac{128}{315}) = \frac{109}{315}$ chance of winning, thus our final answer is $p + q = \boxed{424}$.

Proposed by Eric Shi Chen, Solution by Eric Shi Chen and Benjamin Li

2. **Problem: [16]** An ant is traveling on the coordinate plane. Every second, it moves one unit up, left, right, or down with equal probability. If it starts at $(0,0)$, let $\frac{a}{b}$ be the probability that the ant reaches the point (3,3) at some point in the next 8 seconds. Find $a + b$.

   Solution: To find the probability that the ant reaches $(3,3)$ within 8 seconds, we can find the successful paths based on the possible arrival times $t = 6$ and $t = 8$.

   An ant starting at $(0,0)$ requires a minimum of 6 steps to reach $(3,3)$ (3 moves right, 3 moves up). Since each step changes the coordinate sum $x + y$ by 1, and $3 + 3 = 6$ is even, the ant can only be at $(3,3)$ at even-numbered seconds.

   We can set S as the total possible paths, $4^8 = 65536$. We can also set A be the set of paths reaching $(3,3)$ at $t = 6$, and B be the set of paths reaching $(3,3)$ at $t = 8$.

   We begin by calculating $|A|$. To reach $(3,3)$ at $t = 6$, the first 6 moves must be 3 Right (R) and 3 Up (U). There are $\binom{6}{3} = 20$ such sequences. For each, the remaining 2 steps can be any of 4 directions: $20 \times 4^2 = 320$.

   Next, we calculate $|B|$. For $(3,3)$ at $t = 8$, the moves (n_R, n_L, n_U, n_D) must satisfy $n_R - n_L = 3$ and $n_U - n_D = 3$ with $\sum n_i = 8$. The possible distributions are $(4,1,3,0)$ and $(3,0,4,1)$. Thus,

$$|B| = \frac{8!}{4!1!3!0!} + \frac{8!}{3!0!4!1!} = 280 + 280 = 560.$$

   Now, we calculate the size of their intersection. These paths hit $(3,3)$ at $t = 6$ (20 ways) and return to $(3,3)$ at $t = 8$ (4 ways: RL, LR, UD, DU). $|A \cap B| = 20 \times 4 = 80$.

   Using the Principle of Inclusion-Exclusion, the number of successful paths is:

$$|A \cup B| = 320 + 560 - 80 = 800$$

   The probability is:

$$\frac{800}{65536} = \frac{25}{2048}$$

   This means $a = 25$ and $b = 2048$, so the sum is $a + b = \boxed{2073}$.

   Proposed by Daniel Cai, Solution by Tanish Parida and Anirudh Pulugurtha

3. **Problem: [16]** How many integers from 1 to 2026 divide a number in the form $111\ldots1$ (any number of ones)?

   Solution: Upon observation, it is clear that no multiples of 2 or 5 can divide any of these numbers, since they all end in 1. We now show that every other integer works.

Note that a number consisting of n ones can be written as

$$\frac{10^n - 1}{9}.$$

It suffices to show that for any prime $p \neq 2, 5$, there exists some n such that

$$p \mid \frac{10^n - 1}{9}.$$

Since $p \neq 2, 5$, we have $\gcd(p, 10) = 1$. Consider the sequence

$$10^1, 10^2, 10^3, \ldots \pmod{p}.$$

By Fermat's little theorem, there is an $n = p - 1$ such that

$$10^n - 1 \equiv 0 \pmod{p} \Rightarrow p \mid (10^n - 1).$$

Since $\gcd(p, 9) = 1$ for $p \neq 3$, it follows that

$$p \mid \frac{10^n - 1}{9}.$$

For $p = 3$, we can directly verify that $3 \mid 111$, so it also works.

Thus, every prime $p \neq 2, 5$ divides some number of the form $111\ldots1$, and therefore any integer whose prime factorization does not include 2 or 5 also works.

We now count such integers from 1 to 2026 using inclusion-exclusion:

$$2026 - \left\lfloor \frac{2026}{2} \right\rfloor - \left\lfloor \frac{2026}{5} \right\rfloor + \left\lfloor \frac{2026}{10} \right\rfloor = 2026 - 1013 - 405 + 202 = \boxed{810}.$$

Proposed by Eric Li, Solution by Anirudh Pulugurtha

Round 6

1. **Problem: [18]** There is exactly one real solution to $x^2 + 2x + 1 = \lfloor x^4 + x^3 + x^2 \rfloor$, and this real solution can be expressed as $a + \sqrt{b}$ for integers a and b. Compute $a + b$.

   Solution: First, note that $x^2 + 2x + 1 = (x+1)^2$. Now, consider bounding the possible values of x. Calculating the left and right hand sides for a few small values of x gives the following:

x	$(x+1)^2$	$\lfloor x^4 + x^3 + x^2 \rfloor$
-2	1	12
-1	0	1
0	1	0
1	4	3
2	9	28

   It appears that when $x > 2$ and $x < -1$, the right hand expression grows at a much faster rate than the left hand expression, and thus there will be no real solutions for $x > 2$ and $x < -1$. Additionally, note that $(x+1)^2$ goes from being less than $\lfloor x^4 + x^3 + x^2 \rfloor$ to greater than it somewhere between -1 and 0, and it then goes from being greater than $\lfloor x^4 + x^3 + x^2 \rfloor$ to being less than it between 1 and 2. This implies that there is likely an intersection in one of these two ranges.

   To check for the range where $-1 < x < 0$, notice that due to the floor function, both sides must be integers. However, there are only no integer values of $(x+1)^2$ between $-1 < x < 0$. This is because for $-1 < x < 0$,

$$0 < (x+1)^2 < 1.$$

Thus, the solution cannot lie within this range.

For the range where $1 < x < 2$,

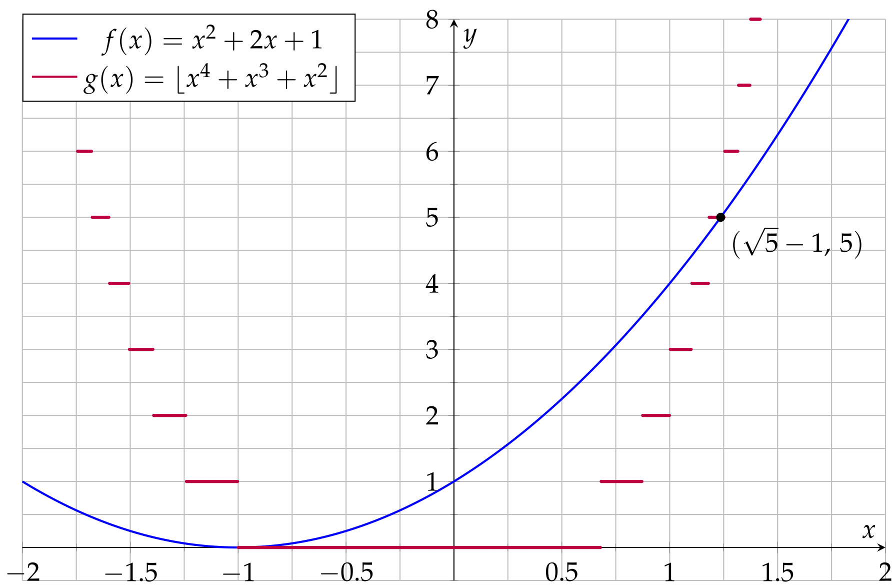
$$4 < (x+1)^2 < 9.$$

Therefore, the possible integer values are 5, 6, 7, and 8. Now, one can simply try all these values and find that if $(x+1)^2 = 5$, then $x = -1+\sqrt{5}$ (since $x > 1$). Then, plugging this into $\lfloor x^4 + x^3 + x^2 \rfloor$ yields

$$\lfloor 1 - 4\sqrt{5} + 30 - 20\sqrt{5} + 25 - 1 + 3\sqrt{5} - 15 + 5\sqrt{5} + 1 - 2\sqrt{5} + 5 \rfloor = \lfloor 46 - 18\sqrt{5} \rfloor.$$

$46 - 18\sqrt{5} \approx 5.75$, so $\lfloor 46 - 18\sqrt{5} \rfloor = 5$. Therefore, the value of x that satisfies the equation is $-1 + \sqrt{5} \Rightarrow a + b = \boxed{4}$.

Remark: Here is a graph showing $f(x) = x^2 + 2x + 1$ and $g(x) = \lfloor x^4 + x^3 + x^2 \rfloor$, with the only intersection point being $(\sqrt{5} - 1, 5)$. Even though it looks like $f(x)$ and $g(x)$ intersect at $(-1, 0)$, the floor function makes it so that there is actually a jump from 0 to 1 at $x = -1$.



Proposed by Daniel Cai, Solution by Nathan Tan

2. **Problem: [18]** Find the sum of all integers n such that $4^n + 105$ is a perfect square.

   Solution: We are solving $4^n + 105 = k^2$ for some n, k. We can rearrange to make calculation simpler:

$$k^2 - 4^n = 105$$

$$k^2 - 2^{2n} = 105$$

Notice that we have two perfect squares on the left hand side; we can factor using the difference of squares:

$$(k - 2^n)(k + 2^n) = 105.$$

From this equation, we know we're looking for factors with a difference of $(k + 2^n) - (k - 2^n) = 2 \cdot 2^n = 2^{n+1}$. We can look at the factor pairings that have a product of 105, looking for differences that are powers of 2.

$$15 \cdot 7 = 105,\ 15 - 7 = 8 = 2^3,\ n = 2$$

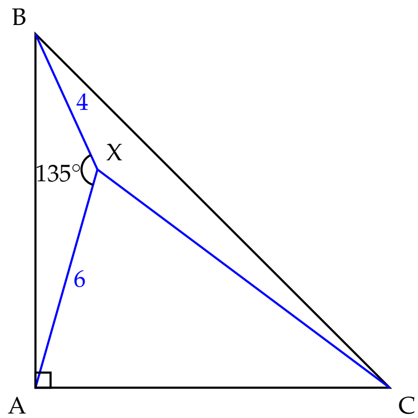
$$21 \cdot 5 = 105,\ 21 - 5 = 16 = 2^4,\ n = 3$$

$$35 \cdot 3 = 105,\ 35 - 3 = 32 = 2^5,\ n = 4$$

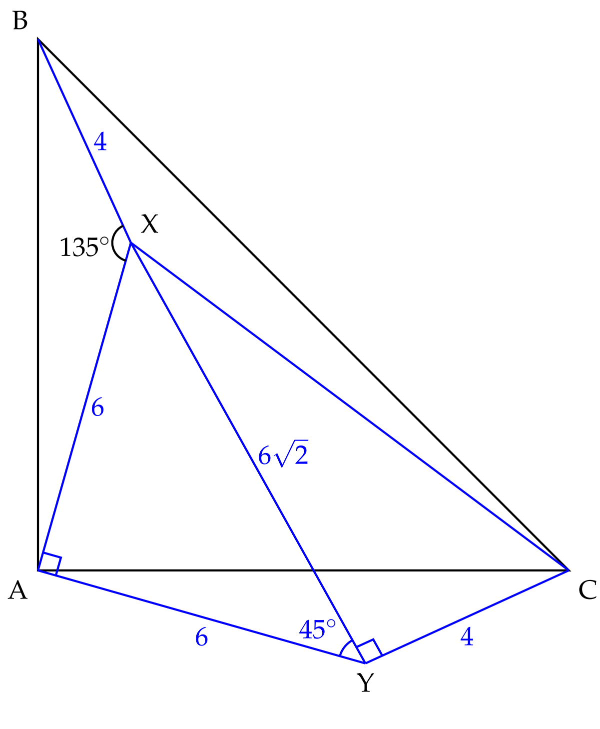
The only values that work are $n = 2, 3, 4$ so the answer is $2 + 3 + 4 = \boxed{9}$.

Proposed by Anirudh Pulugurtha, Solution by Anirudh Pulugurtha

3. **Problem: [18]** In isosceles right triangle ABC with $AB = AC$, X is located in $\triangle ABC$ such that $AX = 6$, $BX = 4$, and $\angle AXB = 135^\circ$. Compute CX^2.



Solution: Consider the following diagram, where we have rotated $\triangle AXB$ 90° clockwise about A.



Notice that B is mapped to C. Suppose X is mapped to the point Y in this rotation. Since $AX = AY$ and AY is AX rotated by 90° with respect to A, we know that $\triangle AXY$ is an isosceles right triangle. Then, by the Pythagorean theorem, we can calculate XY to be $\sqrt{6^2 + 6^2} = 6\sqrt{2}$.

Since rotations preserve angles, we know that $\angle AYC = 135^\circ$, and since $\triangle AXY$ is an isosceles right triangle, we have $\angle XYC = \angle AYC - \angle AYX = 135^\circ - 45^\circ = 90^\circ$. Therefore, $\triangle CYX$ is right, and we can apply the Pythagorean theorem to finish the problem: $CX^2 = CY^2 + XY^2 = (6\sqrt{2})^2 + 4^2 = \boxed{88}$.

Solution 2: Let $AB = AC = s$. Since $\triangle ABC$ is an isosceles right triangle, $\angle A = 90^\circ$. Let $s = AB$. Now, we can apply the Law of Cosines to $\triangle AXB$:

$$\begin{aligned} AB^2 &= AX^2 + BX^2 - 2(AX)(BX)\cos 135^\circ \\ s^2 &= 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot \left(-\frac{\sqrt{2}}{2}\right) \\ s^2 &= 52 + 24\sqrt{2}. \end{aligned}$$

Now let $\theta = \angle BAX$. Then, by law of cosines in $\triangle ABX$,

$$\begin{aligned} BX^2 &= BA^2 + AX^2 - 2(BA)(AX)\cos\theta \\ 16 &= s^2 + 36 - 2 \cdot s \cdot 36 \cdot \cos\theta \\ \cos\theta &= \frac{s^2+20}{12s}. \end{aligned}$$

Because $\angle A = 90^\circ$, we have $\angle XAC = 90^\circ - \theta$. Since θ is positive, we have

$$\begin{aligned} \sin\theta &= \sqrt{1-\cos^2\theta} \\ \sin\theta &= \sqrt{1-\left(\frac{s^2+20}{12s}\right)^2} \\ \sin\theta &= \sqrt{\frac{144s^2-(s^2+20)^2}{144s^2}} \\ \sin\theta &= \sqrt{\frac{-s^4+104s^2-400}{144s^2}} \\ \sin\theta &= \sqrt{\frac{-(52+24\sqrt{2})^2+104(52+24\sqrt{2})-400}{144s^2}} \\ \sin\theta &= \sqrt{\frac{-3856-2496\sqrt{2}+5408+2496\sqrt{2}-400}{144s^2}} \\ \sin\theta &= \sqrt{\frac{1152}{144s^2}} \\ \sin\theta &= \frac{2\sqrt{2}}{s} \end{aligned}$$

We then apply law of cosines to $\triangle AXC$ to find CX^2:

$$\begin{aligned} CX^2 &= AX^2 + AC^2 - 2(AX)(AC)\cos(90^\circ-\theta) \\ CX^2 &= 6^2 + s^2 - 2 \cdot 6 \cdot s \cdot \sin\theta \\ CX^2 &= 36 + 52 + 24\sqrt{2} - 24\sqrt{2} \\ CX^2 &= \boxed{88}. \end{aligned}$$

Proposed by Eric Shi Chen, Solution by Eric Shi Chen

Round 7

1. **Problem: [21]** Suppose

$$A = \sum_{n=1}^{\infty} \frac{3n^2+14n+10}{n(n+1)(n+2)(n+5)}.$$

Calculate $240A$.

Solution: We start by noticing that $3n^2 + 14n + 10 = 2(n+1)(n+5) + n(n+2)$. So, we can rewrite

$$\frac{3n^2+14n+10}{n(n+1)(n+2)(n+5)} = \frac{2}{n(n+2)} + \frac{1}{(n+1)(n+5)} = \frac{1}{n} - \frac{1}{n+2} + \frac{1}{4}\cdot\left(\frac{1}{n+1} - \frac{1}{n+5}\right).$$

Now, we compute

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

and

$$\sum_{n=1}^{\infty}\left[\frac{1}{4}\cdot\left(\frac{1}{n+1} - \frac{1}{n+5}\right)\right] = \frac{1}{4}\cdot\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) = \frac{77}{240}.$$

Therefore,

$$A = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{4}\cdot\left(\frac{1}{n+1} - \frac{1}{n+5}\right)\right] = \frac{3}{2} + \frac{77}{240} = \frac{437}{240},$$

so our final answer is $240A = \boxed{437}$.

Proposed by Eric Shi Chen, Solution by Benjamin Li and Eric Shi Chen

2. **Problem: [21]** Alice has 7 cards, numbered 1 through 7. She builds a deck by flipping a fair coin for each card to decide whether to include it. If she includes no cards, the game ends and she scores 0 points; otherwise, she shuffles the chosen cards and deals them out in a row from left to right.

   Alice then scores points by counting the number of *decreasing runs* in her row of cards. Reading from left to right, a new run begins with the first card, and a new run begins whenever a card's number is strictly greater than the card immediately before it. For example, if her dealt row is $6, 2, 7, 4, 1$, her cards are grouped into two decreasing runs: $(6, 2)$ and $(7, 4, 1)$, so she scores 2 points.

   If the expected number of points Alice scores can be expressed as $\frac{a}{b}$ for relatively prime a, b, compute $a + b$.

   Note: A *fair* coin has a $\frac{1}{2}$ chance of landing on either side.

   Solution: Assuming Alice chooses exactly $k \geq 1$ cards, her score is 1 plus the number of ascents in the dealt row, where an *ascent* is an instance of a card being strictly greater than the one immediately preceding it. Across all $k!$ possible permutations of the chosen cards, the total number of ascents equals the total number of descents by the symmetry of reversing sequences, and since there are exactly $k-1$ adjacent pairs per permutation, the average number of ascents must be $\frac{k-1}{2}$. This symmetry means that the expected number of decreasing runs for a uniformly shuffled deck of k cards evaluates to $1 + \frac{k-1}{2} = \frac{k+1}{2}$.

   Alice flips a fair coin for each of the 7 cards to determine her deck, meaning that the probability of choosing exactly k cards is $\binom{7}{k}\frac{1}{2^7}$. This, in tandem with the fact that an empty deck contributes nothing to the total score, allows us to compute the overall expected number of points by summing the products of the probabilities and conditional expected scores for each $k \in \{1, \dots 7\}$, getting

$$\sum_{k=1}^{7}\binom{7}{k}\frac{1}{2^7}\frac{k+1}{2}$$

We split this summation into two managable parts and then use the binomial theorem to compute:

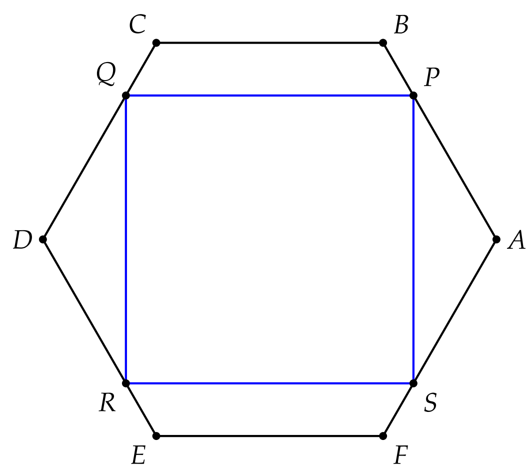
$$\begin{aligned}\sum_{k=1}^{7}\binom{7}{k}\frac{1}{2^7}\frac{k+1}{2} &= \frac{1}{2^8}\sum_{k=1}^{7}k\binom{7}{k}+\frac{1}{2^8}\sum_{k=1}^{7}\binom{7}{k}\\ &= \frac{1}{2^8}\left(7\cdot 2^6\right)+\frac{1}{2^8}\left(2^7-1\right)\\ &= \frac{448+128-1}{256}\\ &= \frac{575}{256}.\end{aligned}$$

Therefore, our final answer is $575+256=\boxed{831}$.

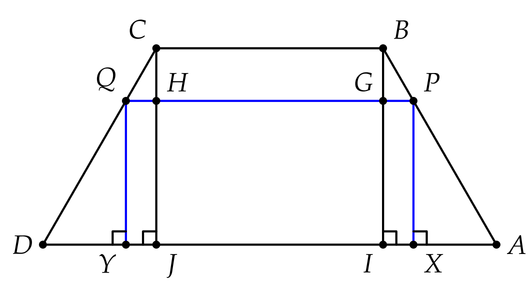
Proposed by Aarush Kulkarni, Solution by Aarush Kulkarni

3. **Problem: [21]** A hexagon H_1 is inscribed in a square S_1. A square S_2 is then inscribed in H_1. This pattern continues indefinitely, with H_n being inscribed inside of S_n and S_{n+1} being inscribed inside of H_n. Suppose the area of H_k is h_k, and the area of S_k is s_k for every positive integer k. Given that $h_1 s_1 = 24\sqrt{3}-41$, the maximum possible value of $(h_1+h_2+h_3+\dots)(s_1+s_2+s_3+\dots)$ can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, c is not divisible by the square of any prime, and $\gcd(a,b,d)=1$. Compute $a+b+c+d$.

   Solution: First, we derive the maximum possible area of a square inscribed inside of a unit hexagon. The configuration below shows the maximum size of such a possible square:



Now, we will zoom in on the top half of the diagram:



In order to maximize the area of the square, we must have $\overline{BC} \parallel \overline{PQ} \parallel \overline{AD}$, and $\overline{AD}$ is the perpendicular bisector of $\overline{PS}$ and $\overline{QR}$. Dropping perpendiculars from B and C to AD, and labeling all of the intersection points as above, we find that the following triangles are all $30-60-90$:

$$\triangle CHQ \cong \triangle BGP \sim \triangle QYD \cong \triangle PXA \sim \triangle CJD \cong \triangle BIA.$$

We are assuming $AB = BC = CD = 1$, and suppose $JY = HQ = IX = GP = a$. Then, we have $DY = DJ - JY = \frac{1}{2} - a$ and thus $QY = DY \cdot \sqrt{3} = \frac{\sqrt{3}}{2} - a\sqrt{3}$. Since AD perpendicularly bisects QY we have $PQ = 2QY$, and we can also derive $PQ = XY = XI + IJ + JY = a + 1 + a = 2a + 1$.

Now, we can now solve for a:

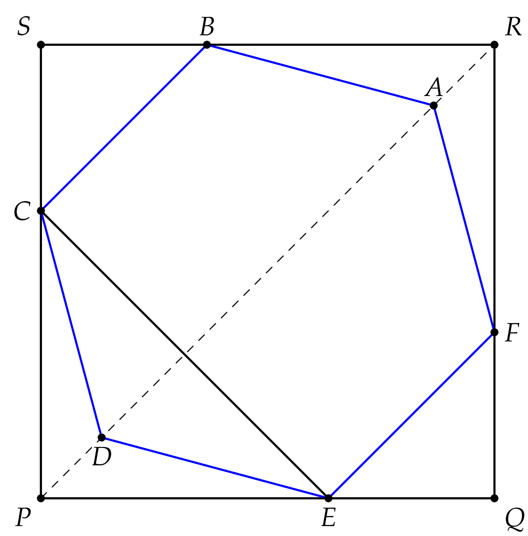
$$\begin{aligned}
PQ &= 2QY \\
2a + 1 &= 2(\tfrac{\sqrt{3}}{2} - a\sqrt{3}) \\
2a + 1 &= \sqrt{3} - 2a\sqrt{3} \\
(2 + 2\sqrt{3})a &= \sqrt{3} - 1 \\
a &= \frac{\sqrt{3}-1}{2+2\sqrt{3}} = \frac{(\sqrt{3}-1)(2\sqrt{3}-2)}{(2\sqrt{3})^2 - 2^2} = \frac{8-4\sqrt{3}}{8} = \frac{2-\sqrt{3}}{2}.
\end{aligned}$$

So, we have the side length of the square is $PQ = GP + HG + QH = 1 + 2a = 3 - \sqrt{3}$.

Thus, we can calculate the ratio of the area of the hexagon to the area of the square to be

$$\begin{aligned}
[ABCDEF] : [PQRS] &= \frac{\sqrt{3}}{4} \cdot 6 \cdot 1^2 : (3-\sqrt{3})^2 \\
&= \frac{3\sqrt{3}}{2} : 12 - 6\sqrt{3} \\
&= 1 : \frac{8\sqrt{3}-12}{3}.
\end{aligned}$$

Next, we derive the maximum possible area of a hexagon inscribed inside of a unit square. The following configuration maximizes the area of the hexagon, where AD is on the diagonal PR:



Suppose $SC = b$. We have $BC = b\sqrt{2}$, which means $CE = BC \cdot \sqrt{3} = b\sqrt{6}$, and thus $CP = \frac{CE}{\sqrt{2}} = b\sqrt{3}$. We can now solve for b: $SC + CP = b + b\sqrt{3} = 1 \implies b = \frac{\sqrt{3}-1}{2}$. Therefore, the side length of the hexagon is $BC = \frac{\sqrt{6}-\sqrt{2}}{2}$.

Now, we can calculate the ratio of the area of square to the area of the hexagon:

$$\begin{aligned}[PQRS] : [ABCDEF] &= 1^2 : \frac{\sqrt{3}}{4} \cdot 6 \cdot \left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 \\ &= 1 : \frac{3\sqrt{3}}{2} \cdot \left(\frac{8-2\sqrt{12}}{4}\right) \\ &= 1 : \frac{3\sqrt{3}\cdot(2-\sqrt{3})}{2} \\ &= 1 : \frac{6\sqrt{3}-9}{2}.\end{aligned}$$

We now know that given s_1, we have $h_1 = \frac{6\sqrt{3}-9}{2}s_1$, $s_2 = \frac{8\sqrt{3}-12}{3}h_1$, and $h_2 = \frac{6\sqrt{3}-9}{2}s_2$. Thus, the ratio $\frac{s_2}{s_1}$ is

$$\frac{s_2}{s_1} = \frac{\frac{8\sqrt{3}-12}{3}h_1}{\frac{2}{6\sqrt{3}-9}h_1} = \frac{4(2\sqrt{3}-3)}{3} \cdot \frac{3(2\sqrt{3}-3)}{2} = 2(2\sqrt{3}-3)^2 = 42 - 24\sqrt{3}.$$

Notice that the ratio $\frac{h_2}{h_1}$ must be the same:

$$\frac{h_2}{h_1} = \frac{\frac{6\sqrt{3}-9}{2}s_2}{\frac{3}{8\sqrt{3}-12}s_2} = \frac{\frac{8\sqrt{3}-12}{3}}{\frac{2}{6\sqrt{3}-9}} = 42 - 24\sqrt{3}.$$

Therefore, we have $s_k = s_1 \cdot (42 - 24\sqrt{3})^{k-1}$ and $h_k = h_1 \cdot (42-24\sqrt{3})^{k-1}$.
The sum of an infinite geometric series with a ratio of $r < 1$ and starting term a is $\frac{a}{1-r}$, so we have

$$(h_1 + h_2 + h_3 + \dots)(s_1 + s_2 + s_3 + \dots) = \frac{h_1 s_1}{[1-(42-24\sqrt{3})]^2} = \frac{24\sqrt{3}-41}{(24\sqrt{3}-41)^2} = \frac{41+24\sqrt{3}}{47}.$$

Thus, our final answer is $41 + 24 + 3 + 47 = \boxed{115}$.

Proposed by Eric Shi Chen, Solution by Eric Shi Chen

Estimation

Problem: Let the *Anirudhian operation*, denoted $\Lambda(n)$, be defined as the product of every prime $p \leq n$: in particular, $\Lambda(n) = \prod_{p \leq n} p$. Using this information, estimate the number of digits of

$$\frac{2026!}{\Lambda(2026)}$$

in base 10. You will earn $\lfloor 20 \min(A/E, E/A)^4 \rfloor$ points, where E is your estimate, and A is the actual answer.

Solution: In this solution, log denotes the natural logarithm, and we set $n := 2026$. The number of digits of a large integer X is approximately $\log_{10}(X)$, and so by change of base, we want to estimate

$$\log_{10}\left(\frac{n!}{\Lambda(n)}\right) = \frac{\log(n!) - \log\Lambda(n)}{\log 10}.$$

First, we estimate $\log\Lambda(n)$ by taking the logarithm in order to turn the product into a sum. From the prime number theorem, a positive integer x as a "probability" of $\frac{1}{\log x}$ of being prime. This, along with estimating this sum with an integral, gets

$$\log\Lambda(n) = \sum_{p \leq n} \log p \approx \int_2^n (\log x) \cdot \frac{1}{\log x}\, dx = \int_2^n 1\, dx \approx n.$$

Next, we estimate $\log(n!)$: from Stirling's approximation, we have

$$\log(n!) \approx \log\left(\left(\frac{n}{e}\right)^n\right) = n(\log n - 1).$$

Thus, our natural log estimate simplifies nicely to

$$\log(n!) - \log \Lambda(n) \approx n(\log n - 1) - n = n(\log n - 2).$$

Now we just compute the values: We can quickly estimate $\log n$ by rounding to the nearest power of 2

$$\log n \approx \log 2048 = 11 \log 2 \approx 11 \cdot 0.69 = 7.59,$$

and plugging this in along with $\log 10 \approx 2.30$, our base 10 estimate becomes

$$\frac{n(\log n - 2)}{\log 10} \approx \frac{2026(7.59 - 2)}{2.30} = \frac{2026 \cdot 5.59}{2.30} = \frac{11325}{2.3} = \frac{113250}{23} \approx 4924,$$

which gets a nice 19 points, as the exact answer is 4970.

Remark: Notice that we dropped the $\sqrt{2\pi n}$ term from Stirling's approximation. This looks terrible, but it's all inside a log, so the dropped term is relatively tiny compared to n.

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