Acton-Boxborough Math Competition Online Contest Solutions

Saturday, December 13 — Sunday, December 14, 2025

1. **Problem:** Compute the value of $-\dfrac{2\cdot 0+2\cdot 5}{2-0+2-5}$.

   Answer: $\boxed{10}$

   Solution: $-\dfrac{2\cdot 0+2\cdot 5}{2-0+2-5} = -\dfrac{10}{-1} = -(-10) = \boxed{10}$.

   Proposed by Anirudh Pulugurtha and Eric Shi Chen, Solution by Jonathan Ren

2. **Problem:** A square has a perimeter of 16. A rectangle with the same perimeter as the square has side lengths in a ratio of 3 : 1. Compute the area of the rectangle.

   Answer: $\boxed{12}$

   Solution: First, the side length of the square is $16/4 = 4$. Therefore, we can set the length of the rectangle to be x and the width to be $3x$ (because of the 3 : 1 ratio between the sides). The perimeter of the rectangle is 16, so $2(3x + x) = 8x = 16 \Rightarrow x = 2$. Thus, the area is $2 \cdot 2(3) = \boxed{12}$.

   Proposed by Anirudh Pulugurtha, Solution by Aarush Kulkarni

3. **Problem:** A bag has cards labeled 1, 2, 3, 4, and 5. If two cards are randomly selected without replacement, the probability that the sum is even can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

   Answer: $\boxed{7}$

   Solution: The total number of ways to choose 2 cards from 5 without replacement is $\binom{5}{2} = 10$. The sum of two integers is even if and only if both integers are even or both integers are odd. The set of numbers is $\{1,2,3,4,5\}$, which contains 3 odd numbers, $\{1,3,5\}$, and 2 even numbers, $\{2,4\}$. We therefore have two cases:

   Case 1: Both cards are odd. The number of ways to choose 2 odd cards from 3 is $\binom{3}{2} = 3$.

   Case 2: Both cards are even. The number of ways to choose 2 even cards from 2 is $\binom{2}{2} = 1$.

   The total number of favorable outcomes is $3+1 = 4$; thus, the probability is $\frac{4}{10} = \frac{2}{5}$, and so the answer is $2+5 = \boxed{7}$.

   Proposed by Eric S Chen, Solution by Aarush Kulkarni

4. **Problem:** What is the 100th digit to the right of the decimal point of $\frac{12}{99}$?

   Answer: $\boxed{2}$

   Solution: The decimal expansion of $ab/99$ is $0.\overline{ab}$, where a and b are arbitrary digits, and thus we have

   $$\frac{12}{99} = \frac{4}{33} = 0.\overline{12},$$

   where the digits repeat with a period of 2. In particular, the n^{th} digit is given by

   $$\begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}.$$

   Since 100 is a even number, the 100^{th} digit is $\boxed{2}$.

   Proposed by Nathan Tan, Solution by Aarush Kulkarni

5. **Problem:** Kevin and Eric are bike racing. Eric gives Kevin a 2-mile and 30-minute head start. If Kevin bikes at 15 miles per hour and Eric bikes at 20 miles per hour, how many minutes would Eric take to catch up to Kevin?

   Answer: $\boxed{114}$

   Solution: We make use of $d = r \cdot t$, which relates distance to rate and time.

   Kevin starts with a head start of 2 miles plus the distance covered in 30 minutes. Since 30 minutes is 0.5 hours, the distance Kevin travels in this time is $15 \times 0.5 = 7.5$ miles, and so the total initial distance between Eric and Kevin when Eric starts is $d = 2 + 7.5 = 9.5$ miles.

   Let t be the time in hours Eric takes to catch up. Since Eric must close the gap at a relative speed of $20 - 15$, we have, $t = \frac{d}{20-15} = \frac{9.5}{5} = 1.9$ hours, or $1.9 \times 60 = \boxed{114}$ minutes.

   Proposed by Daniel Cai, Solution by Aarush Kulkarni

6. **Problem:** The product of all factors of 2025 can be written as $a \cdot 2025^b$, where a and b are integers and $a < 2025$. Find $a + b$.

   Answer: $\boxed{52}$

   Solution: We begin by finding the prime factorization of 2025: $2025 = 45^2 = (9 \cdot 5)^2 = (3^2 \cdot 5)^2 = 3^4 \cdot 5^2$ So, the number of divisors is $(4+1)(2+1) = 5 \cdot 3 = 15$. Note that by pairing up divisors that multiply to 2025, we have 2025^7, along with a singular 45 due to the fact that 2025 is a perfect square. Thus, the product of all the divisors is $2025^7 \cdot 45 = 45^{14} \cdot 45 = 45^{15}$.

   We wish to write this in the form $a \cdot 2025^b$. We manipulate the expression as follows, $45^{15} = 45 \cdot 45^{14} = 45 \cdot (45^2)^7 = 45 \cdot 2025^7$. This satisfies the problem conditions, so $a + b = 45 + 7 = \boxed{52}$.

   Proposed by Jonathan Ren, Solution by Aarush Kulkarni

7. **Problem:** Triangle ABC is an isosceles triangle with $AB = AC = 10$, and height AD has length 6. The median from B intersects AC at the point M, and BM intersects AD at point N. If $\triangle BCN$ is rotated 360° around the line BC, the volume of the resulting two cones formed can be expressed as $\frac{a\pi}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

   Answer: $\boxed{67}$

   Solution: When $\triangle BCN$ is rotated around BC, the two cones formed are congruent as the isosceles triangle is symmetric about height AD. The radii of the bases of both cones is then ND. Note that AD is a median in addition to BM, so the intersection point, N, must be the centroid of $\triangle ABC$.

   The centroid splits every median by a $2:1$ ratio. Thus, we have $AN : ND = 2 : 1$. Since $AN + ND = 6$, this results in $AN = 4, ND = 2$ and the radius of each cone is 2. Additionally, the height from the vertex of an isosceles triangle bisects the base, so since $BC = 16$, $BD = DC = 8$. This means the height of each cone is 8.

   Now, using the formula for the volume of a cone, we find that each cone has a volume of

   $$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2^2)(8) = \frac{32}{3}\pi.$$

   This means that the combined volume is

   $$2 \cdot \frac{32}{3}\pi = \frac{64}{3}\pi,$$

   and thus the answer is $a + b = 64 + 3 = \boxed{67}$.

   Proposed by Raymond Gao, Solution by Nathan Tan

8. **Problem:** How many positive integer divisors of 20^{25} are perfect cubes but not perfect squares?

Answer: $\boxed{108}$

Solution: The prime factorization of 20^{25} is $(2^2 \cdot 5)^{25} = 2^{50} \cdot 5^{25}$. A divisor of 20^{25} is of the form $2^x \cdot 5^y$, where $0 \leqslant x \leqslant 50$ and $0 \leqslant y \leqslant 25$.

For a divisor to be a perfect cube, the exponents x and y must be multiples of 3. The possible values for x are $\{0, 3, 6, \dots, 48\}$, which contains $\lfloor \frac{50}{3} \rfloor + 1 = 17$ values; the possible values for y are $\{0, 3, 6, \dots, 24\}$, which contains $\lfloor \frac{25}{3} \rfloor + 1 = 9$ values. Thus, the total number of perfect cube divisors is $17 \cdot 9 = 153$.

We must exclude those divisors that are also perfect squares: a number is both a perfect cube and a perfect square if and only if it is a perfect 6th power. This requires x and y to be multiples of 6. The possible values for x are $\{0, 6, 12, \dots, 48\}$, which contains $\lfloor \frac{50}{6} \rfloor + 1 = 9$ values; the possible values for y are $\{0, 6, 12, \dots, 24\}$, which contains $\lfloor \frac{25}{6} \rfloor + 1 = 5$ values. Thus, the total number of divisors that are perfect 6th powers is $9 \cdot 5 = 45$.

Subtracting the perfect 6th powers from the perfect cubes, we find

$$153 - 45 = \boxed{108}.$$

Proposed by Eric Shi Chen, Solution by Aarush Kulkarni

9. **Problem:** Let T_n be the n^{th} triangular number defined by $T_n = 1 + 2 + 3 + \cdots + n$. If $\tau_n = T_1 + T_2 + T_3 + \cdots + T_n$ compute the value of $\tau_1 + \tau_2 + \tau_3 + \cdots + \tau_{25}$.

Answer: $\boxed{20475}$

Solution: Since T_n is the sum of the first n positive integers, $T_n = \dfrac{n(n+1)}{2} = \dbinom{n+1}{2}$.

By the Hockey-stick identity, $\displaystyle\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1}$. Thus,

$$\tau_n = \sum_{i=1}^{n} T_n = \sum_{i=1}^{n} \binom{i+1}{2} = \binom{n+2}{3}.$$

Summing over $\tau_1, \tau_2, \tau_3, \dots, \tau_{25}$ and applying Hockey-stick again, we find

$$\tau_1 + \tau_2 + \tau_3 + \cdots + \tau_{25} = \sum_{n=1}^{25} \tau_n = \sum_{n=1}^{25} \binom{n+2}{3} = \binom{28}{4} = \frac{28 \cdot 27 \cdot 26 \cdot 25}{4 \cdot 3 \cdot 2 \cdot 1} = \boxed{20475}.$$

Proposed by Nathan Tan, Solution by Aarush Kulkarni

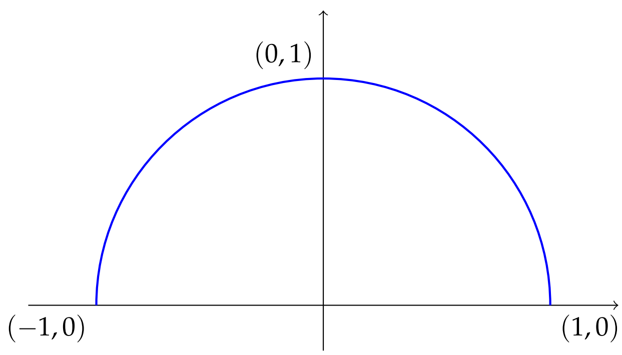
10. **Problem:** Draw a semi-circle with the equation $f(x) = \sqrt{1 - x^2}$. Each second, $f(x)$ will undergo one of the following transformations with a 1/5th probability:

- Do nothing.
- Rotate clockwise 90°.
- Rotate counterclockwise 90°.
- Double the radius of the semicircle.
- Halve the radius of the semicircle.

The probability that $f(x)$ will be at its original position after 5 seconds can be expressed as $\frac{p}{q}$, where p and q are relatively prime. Find $p+q$.

Answer: $\boxed{3356}$

Solution: Here is the graph of the semi-circle in question:



We want this semicircle to return to its original position after 5 seconds. This require both of the following conditions to be true:

- The net rotation is 0°.
- The net scaling factor is 1.

Since there are 5 equally likely transformations each second, the total number of possible outcomes is $5^5 = 3125$.

Now, we break this into cases based on how many times the radius is doubled and halved.

Case 1: 0 doubles and halves. All 5 transformations must be chosen from

$$(\text{do nothing}),\ (\text{rotate clockwise } 90^\circ),\ (\text{rotate counterclockwise } 90^\circ).$$

Notice that either (rotate clockwise 90°) and (rotate counterclockwise 90°) occur the same number of times or one of them occurs 4 times. Let A represent (do nothing), B represent (rotate clockwise 90°), and C represent (rotate counterclockwise 90°). We will split this case into subcases for calculation:

Subcase A: 5 (do nothing)'s. There is only 1 way to order the 5 (do nothing)'s.

Subcase B: 3 (do nothing)'s, 1 (rotate clockwise 90°) and 1 (rotate counterclockwise 90°). This is equivalent to arranging the letters $AAABC$, and there are $\frac{5!}{3!1!1!} = 20$ ways to do so.

Subcase C: 1 (do nothing), 2 (rotate clockwise 90°)'s and 2 (rotate counterclockwise 90°)'s. This is equivalent to arranging the letters $ABBCC$, and there are $\frac{5!}{1!2!2!} = 30$ ways to do so.

Subcase D: 1 (do nothing), 4 (rotate clockwise 90°)'s. This is equivalent to arranging the letters $ABBBB$, and there are $\frac{5!}{1!4!} = 5$ ways to do so.

Subcase E: 1 (do nothing), 4 (rotate counterclockwise 90°)'s. This is equivalent to arranging the letters $ACCCC$, and there are $\frac{5!}{1!4!} = 5$ ways to do so.

Summing these subcases yields a total of $1+20+30+5+5 = 61$ ways for $f(x)$ to return to its original position.

Case 2: 1 double and 1 half. The remaining 3 transformations must be chosen from

$$(\text{do nothing}),\ (\text{rotate clockwise } 90^\circ),\ (\text{rotate counterclockwise } 90^\circ).$$

We proceed with more casework, letting H represent (Halve the radius) and D represent (Double the radius).

Subcase A: 3 (do nothing)'s. This is equivalent to arranging the letters $HDAAA$, and there are $\frac{5!}{1!1!3!} = 30$ ways to do so.

Subcase B: 1 (do nothing), 1 (rotate clockwise 90°) and 1 (rotate counterclockwise 90°). This is equivalent to arranging the letters $HDABC$, and there are $5! = 120$ ways to do so.

Summing these subcases yields a total of $20 + 120 = 140$ ways for $f(x)$ to return to its original position.

Case 3: 2 doubles and 2 halves. Notice that the final transformation can only be (do nothing). This is equivalent to arranging the letters $HHDDA$, and there are $\frac{5!}{2!2!1!} = 30$ ways of doing so.

Adding these 3 cases yields $61 + 140 + 30 = 231$ successful sequences. So, the probability that $f(x)$ will be at its original position is $\frac{231}{3125}$, and thus $p + q = 231 + 3125 = \boxed{3356}$.

Proposed by Daniel Ren, Solution by Eric Shi Chen

11. **Problem:** Bob has three unfair coins with the following probabilities of landing on heads:

- Coin 1: $\frac{6}{7}$
- Coin 2: $\frac{3}{4}$
- Coin 3: $\frac{1}{3}$

Bob flips all three coins simultaneously and observes that the total number of heads is at least 2. He then flips all three coins again. Given this observation, the probability that the second flip yields the same total number of heads as the first flip can be expressed as $\frac{a}{b}$ for relativity prime positive integers a and b. Compute $a + b$.

Answer: $\boxed{283}$

Solution: First, we calculate the probability of 2 or 3 coins landing on heads.

Case 1: 2 coins land on heads. Either the 1st, 2nd, and 3rd coin must land on tails. We can calculate each of these cases separately and sum to get

$$\frac{1}{7} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{6}{7} \cdot \frac{1}{4} \cdot \frac{1}{3} + \frac{6}{7} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{15}{28}.$$

Case 2: 3 coins land on heads. All 3 coins must land on heads (HHH): $\frac{6}{7} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{3}{14}$.

So, the observation that at least 2 coins have landed on heads occurs with a $\frac{15}{28} + \frac{3}{14} = \frac{21}{28} = \frac{3}{4}$.

The probability that both flips yield 2 heads is $\left(\frac{15}{28}\right)^2$, and the probability that both flips yield 3 heads is $\left(\frac{3}{14}\right)^2$. So, the probability that both flips yield the same number of heads given that at least 2 coins have landed on heads is $\dfrac{\left(\frac{15}{28}\right)^2 + \left(\frac{3}{14}\right)^2}{\frac{3}{4}} = \dfrac{\frac{225+36}{784}}{\frac{3}{4}} = \frac{261}{784} \cdot \frac{4}{3} = \frac{87}{196}$.

Thus, $a + b = 87 + 196 = \boxed{283}$.

Proposed by Eric Shi Chen, Solution by Eric Shi Chen

12. **Problem:** Let $f(x)$ be a 5th degree polynomial with the constant term being 5. For $1 \leqslant n \leqslant 5$, let $f(n) = (10 - n)f(0)$. Find $(f(6))^2$.

Answer: $\boxed{4225}$

Solution: We are given that the constant term of $f(x)$ is 5, which implies $f(0) = 5$. For each $n \in \{1, \dots 5\}$, we have the condition

$$f(n) = (10 - n)f(0) = 5(10 - n).$$

Define the auxiliary polynomial $g(x) := f(x) - 5(10 - x)$. Since $\deg f = 5$, it is clear that $\deg g \leqslant 5$. The condition on $f(n)$ implies that $g(n) = 0$ for all $n \in \{1, \dots, 5\}$. Consequently, we may write

$$g(x) = c(x-1)(x-2)(x-3)(x-4)(x-5)$$

for some constant c. Evaluating at $x = 0$, we find that

$$g(0) = f(0) - 5(10) = 5 - 50 = -45.$$

Using the factored form, we obtain

$$g(0) = c(-1)(-2)(-3)(-4)(-5) = -120c.$$

This implies that $-120c = -45$, so $c = \frac{3}{8}$. We are now in a position to compute $f(6)$. First, we evaluate $g(6)$ to get

$$g(6) = \frac{3}{8}(6-1)(6-2)(6-3)(6-4)(6-5) = \frac{3}{8}(120) = 45.$$

Thus,

$$f(6) = g(6) + 5(10 - 6) = 45 + 20 = 65,$$

and thus the answer is $(f(6))^2 = 65^2 = \boxed{4225}$.

Alternate Solution 1 (Advanced): Alternatively, we can use the *Lagrange interpolation formula*. Since $f(x)$ is a polynomial of degree 5, it is uniquely determined by its values at $x \in \{0, \dots, 5\}$. We are given the values $f(0) = 5, f(1) = 45, f(2) = 40, f(3) = 35, f(4) = 30$, and $f(5) = 25$. The value at $x = 6$ is given by

$$f(6) = \sum_{j=0}^{5} f(j)L_j(6),$$

where $L_j(x) = \prod_{k \neq j} \frac{x-k}{j-k}$.

Calculating the terms individually, we have

$$f(0)L_0(6) = 5 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{-1 \cdot -2 \cdot -3 \cdot -4 \cdot -5} = 5(-1) = -5$$
$$f(1)L_1(6) = 45 \cdot \frac{6 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot -1 \cdot -2 \cdot -3 \cdot -4} = 45(6) = 270$$
$$f(2)L_2(6) = 40 \cdot \frac{6 \cdot 5 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot -1 \cdot -2 \cdot -3} = 40(-15) = -600$$
$$f(3)L_3(6) = 35 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot -1 \cdot -2} = 35(20) = 700$$
$$f(4)L_4(6) = 30 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot -1} = 30(-15) = -450$$
$$f(5)L_5(6) = 25 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 25(6) = 150$$

Summing these terms gets $f(6) = -5 + 270 - 600 + 700 - 450 + 150 = 65$. Thus, $(f(6))^2 = \boxed{4225}$.

Remark: Basically the Lagrange interpolation gives a formula for the unique polynomial of degree $\leqslant n$ that passes through $n+1$ points. Here, $f(x)$ is a degree 5 polynomial, so it is completely determined by its values at 6 points. We are given:

$$f(0), f(1), f(2), f(3), f(4), f(5),$$

so $f(x)$ is uniquely determined. The general formula for any polynomial $g(x)$ is

$$g(x) = \sum_{i=0}^{n} g(x_i)\, L_i(x),$$

where each $L_i(x)$ is the *Lagrange basis polynomial*. Each Lagrange basis polynomial $L_i(x)$ is defined as:

$$L_i(x) = \prod_{\substack{0 \leqslant j \leqslant n \\ j \neq i}} \frac{x - x_j}{x_i - x_j}.$$

This ensures that

$$L_i(x_i) = 1 \quad \text{and} \quad L_i(x_j) = 0 \text{ for all } j \neq i.$$

Multiplying $L_i(x)$ by $g(x_i)$ ensures that the polynomial matches the value $g(x_i)$ at x_i. Summing over all i combines all contributions, producing a polynomial that passes through all the points:

$$g(x) = g(x_0)L_0(x) + g(x_1)L_1(x) + \cdots + g(x_n)L_n(x).$$

In this problem, we have $n = 5$, and $x_0 = 0, x_1 = 1, ..., x_5 = 5$. Plugging in $x = 6$ gives:

$$f(6) = \sum_{j=0}^{5} f(j) \prod_{\substack{0 \leqslant k \leqslant 5 \\ k \neq j}} \frac{6-k}{j-k},$$

which matches with $f(6) = \sum_{j=0}^{5} f(j)L_j(6)$, where $L_j(x) = \prod_{k \neq j} \frac{x-k}{j-k}$, as stated in the above solution.

Alternate Solution 2 (Advanced): We employ the method of finite differences. Let Δ denote the *forward difference operator*, defined by $\Delta f(x) := f(x+1) - f(x)$. Since $\deg f = 5$, it follows that $\Delta^6 f(0) = 0$. Using the expansion $\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k)$, we have

$$\sum_{k=0}^{6} (-1)^{6-k} \binom{6}{k} f(k) = 0.$$

Solving for $f(6)$ gets

$$f(6) = -\sum_{k=0}^{5} (-1)^{6-k} \binom{6}{k} f(k) = \sum_{k=0}^{5} (-1)^{7-k} \binom{6}{k} f(k).$$

Substituting the values which we can compute by hand, we have

$$\begin{aligned} f(6) &= 6f(5) - 15f(4) + 20f(3) - 15f(2) + 6f(1) - f(0) \\ &= 6(25) - 15(30) + 20(35) - 15(40) + 6(45) - 5 \\ &= 150 - 450 + 700 - 600 + 270 - 5 \\ &= 65, \end{aligned}$$

and thus $(f(6))^2 = \boxed{4225}$.

Proposed by Eric Shi Chen, Solution by Eric Shi Chen and Aarush Kulkarni

13. **Problem:** There is a right triangular prism $ABCC'A'B'$ with bases $ABC \cong A'B'C'$. $AB = 20$, $AC = 24$, and $BC = 18$. There is a point M on AB such that $AM = 9$ and $BM = 11$. There is a point N on $B'C'$ such that BN is parallel to the plane of $\triangle A'MC$. Given that the length $B'N$ can be expressed as $\frac{a}{b}$ for relativity prime positive integers a and b. Compute $a + b$.

Answer: $\boxed{173}$

Solution: Let D be a point on the same plane as $\triangle ABC$ such that $MBDC$ is a parallelogram with $\overline{MB} \parallel \overline{CD}$ and $\overline{MC} \parallel \overline{BD}$. Let M' be a point on $\overline{A'B'}$ such that $A'M' = 11$ and $M'B = 9$.

Notice that $MBDC$, $MBM'A'$, and $CDM'A'$ are all parallelograms, so $\triangle A'MC$ and $\triangle M'BD$ must be parallel (and congruent). Therefore, $\overline{BN}$ must lie on the same plane as $\triangle M'BD$ because they are both parallel to $\triangle A'MC$ and pass through point B.

Therefore, we must have $\overline{MC} \parallel \overline{M'N}$, $\overline{MB} \parallel \overline{M'B'}$, and $\overline{BC} \parallel \overline{B'N}$, which means that $\triangle MBC \sim \triangle M'B'N$.

We know that $MB = 11$, $BC = 18$, and $M'B' = 9$, so after using the similarity $\frac{MB}{BC} = \frac{M'B'}{B'N}$ we can determine that $\frac{11}{18} = \frac{9}{B'N}$. Thus, $B'N = \frac{162}{11}$, and we see that $a + b = 162 + 11 = \boxed{173}$.

Proposed by Benjamin Li, Solution by Eric Li

14. **Problem:** Suppose a and b are randomly chosen numbers such that $0 < a < 6$ and $0 < b < 7$. Let $f(n)$ denote the largest integer k such that $3^k \leqslant n$. The probability that $f(a) = f(b)$ can be expressed as $\frac{x}{y}$ where x and y are relatively prime positive integers. Find $x + y$.

Answer: $\boxed{39}$

Solution:

Observe that if $f(a) = f(b) = k$, then both a and b must fall into the same interval $[3^k, 3^{k+1})$. Because a ranges over $(0, 6)$ and b ranges over $(0, 7)$, k is at most 1.

We now proceed with casework:

Case 1: $f(a) = f(b) = 1$. Since $0 < a < 6$ and $0 < b < 7$, $f(a) = 1$ when $3 \leqslant a < 6$ and $f(b) = 1$ if when $3 \leqslant b < 7$. The probability that both of these occur is $\frac{3}{6} \cdot \frac{4}{7} = \frac{2}{7}$.

Case 2: $f(a) = f(b) = 0$. In this case, $1 \leqslant a, b < 3$. This occurs with probability $\frac{2}{6} \cdot \frac{2}{7} = \frac{2}{21}$.

Case 3: $f(a) = f(b) = -1$. In this case, $\frac{1}{3} \leqslant a, b < 1$. This occurs with probability $\frac{\frac{2}{3}}{6} \cdot \frac{\frac{2}{3}}{7} = \frac{2}{21} \cdot \frac{1}{9}$.

Notice that we can generalize cases 2 and onward. If $f(a) = f(b) = -n$, where n is a nonnegative integer, then $\frac{1}{3^n} \leqslant a, b < \frac{1}{3^{n-1}}$. (We can safely claim this because these intervals always lie in $(0, 1)$, which is in the range of both a and b). The probability that $f(a) = f(b) = -n$ given any such n is $\frac{\frac{2}{3^n}}{6} \cdot \frac{\frac{2}{3^n}}{7} = \frac{2}{21} \cdot \left(\frac{1}{9}\right)^n$.

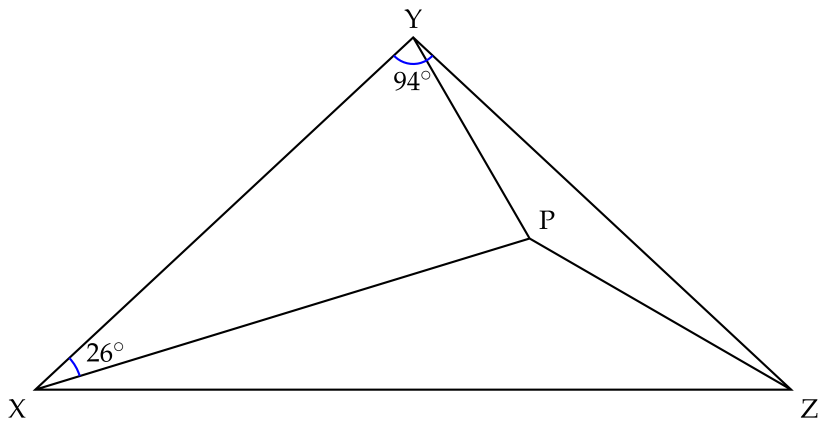
Thus, the sum of cases 2 and onward is simply the sum of the infinite geometric series

$$\frac{2}{21} + \frac{2}{21} \cdot \frac{1}{9} + \cdots + \frac{2}{21} \cdot \left(\frac{1}{9}\right)^n + \cdots = \frac{\frac{2}{21}}{1 - \frac{1}{9}} = \frac{3}{28}.$$

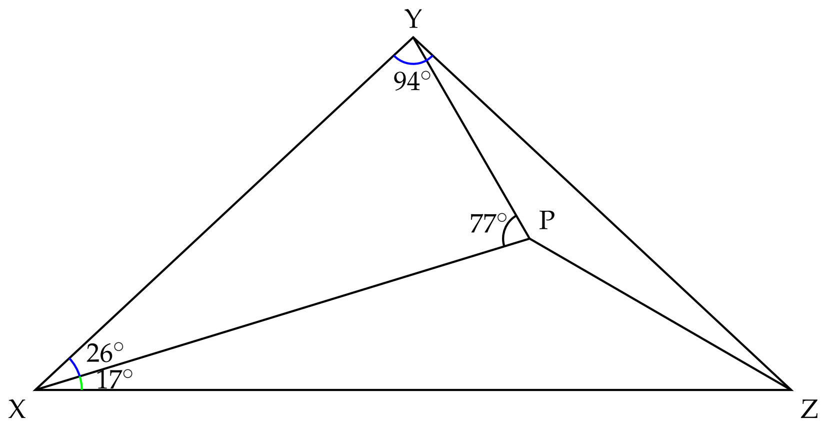
Adding Case 1, our total probability is $\frac{2}{7} + \frac{3}{28} = \frac{11}{28}$, so $x + y = 11 + 28 = \boxed{39}$.

Proposed by Eric Shi Chen, Solution by Christina Lu

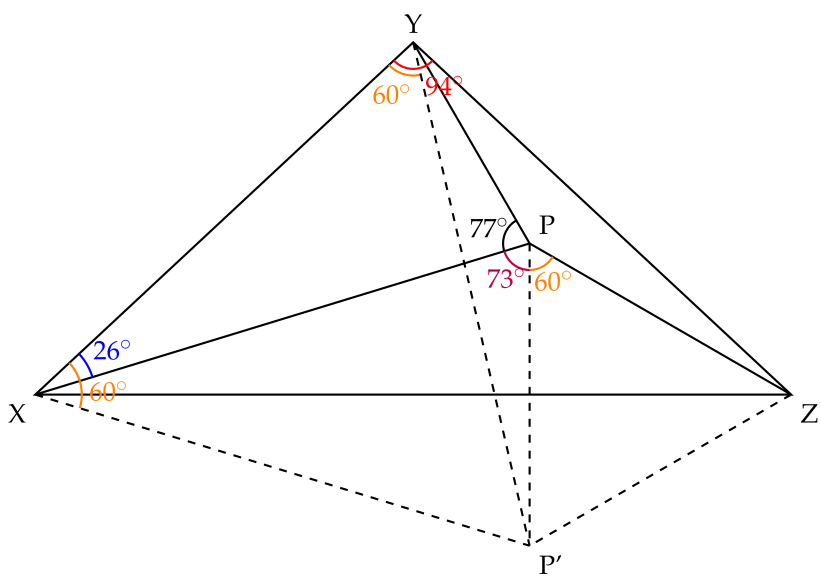
15. **Problem:** In isosceles triangle XYZ, $XY = YZ$ and $\angle XYZ = 94^\circ$. P is a point inside $\triangle ABC$ such that $\triangle XYP$ is isosceles and $\angle PXY = 26^\circ$. Find the measure of $\angle XPZ$ in degrees.



Answer: $\boxed{133}$
Solution:



First, we derive that $\angle PXZ = \angle YXZ - \angle PXY = \dfrac{180^\circ - 94^\circ}{2} - 26^\circ = 17^\circ$ and $\angle XYP = \angle XPY = \dfrac{180^\circ - 26^\circ}{2} = 77^\circ$. Now, we will reflect P over the line XZ to P'.



Notice that $\angle PXZ = \angle P'XZ = 17^\circ$, $\angle YXP' = 26^\circ + 17^\circ + 17^\circ = 60^\circ$, and $XY = XP = XP'$ since $\triangle XYP$ is isosceles and reflection preserves length. Thus $\triangle XYP'$ is equilateral.

We now have 5 segments of equal length:

$$XY = XP = XP' = YP' = YZ$$

Next, we find that $\angle PYZ = \angle XYZ - \angle XYP = 94^\circ - 77^\circ = 17^\circ$, and $\angle PYP' = \angle XYP - \angle XYP' = 77^\circ - 60^\circ = 17^\circ$, so $\triangle YPP'$ is congruent to $\triangle YPZ$ by SAS.

Now, notice that $PZ = P'Z = PP'$, thus $\triangle ZPP'$ is equilateral, and $\angle P'PZ = 60^\circ$. Since $XPP' = 90^\circ - 17^\circ = 73^\circ$. Solving for $\angle XPZ$ yields $\angle XPZ = \angle XPP' + \angle P'PZ = 73^\circ + 60^\circ = \boxed{133}$ degrees.

Proposed by Eric Shi Chen, Solution by Eric Shi Chen