Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 18 — Sunday, October 19, 2025

1. **Problem:** Compute $(2^0+2^6)-(2^0+2^5)$.

   Answer: $\boxed{32}$

   Solution: Recall that the result of raising a number to the power of 0 is just 1. Thus, we have $(1+64)-(1+32)=\boxed{32}$.

   Proposed by Aarush Kulkarni, Solution by Aarush Kulkarni

2. **Problem:** Let $S=2026^2$, find the product of the non-zero digits of S.

   Answer: $\boxed{4032}$

   Solution: Of course, we can always compute computing $2026^2=4104676$, giving us the answer of $4\cdot1\cdot4\cdot6\cdot7\cdot6=4032$.

   A more elegant approach would be to write 2026^2 as

   $$(2000+26)^2=2000^2+2\cdot2000\cdot26+26^2=4\cdot10^6+1.04\cdot10^5+676$$

   Note that, for the sake of clarity, the non-zero digits in the first term are in the 10^6-th place, the non-zero digits in the second term are in the 10^5-th and 10^3-th places, and the non-zero digits in the last term are in the 100s and 10s, and 1s places. Since no two of these non-zero digits occupy the same place-value, we know that each of them actually appear in the expansion of 2026^2, and vice versa.

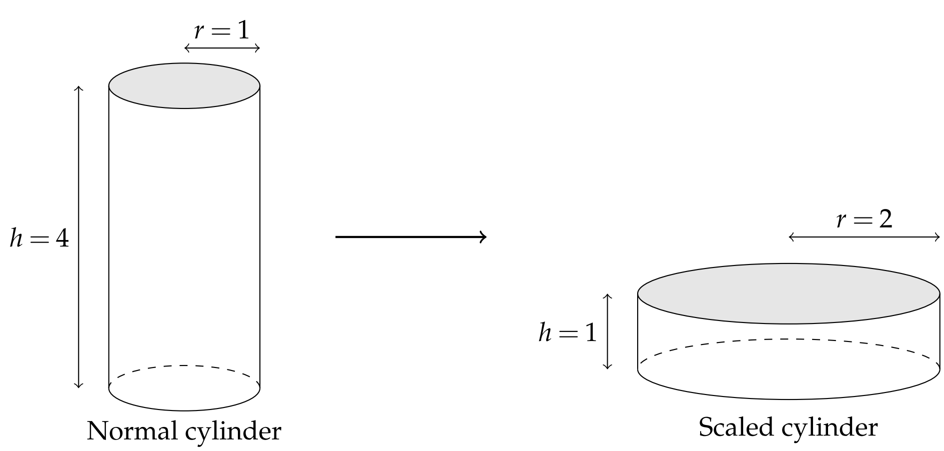
   Hence, our answer is, using the established $4\cdot10^6+1.04\cdot10^5+676$,

   $$4\cdot1\cdot4\cdot6\cdot7\cdot6=\boxed{4032}.$$

   Proposed by Steven Feng, Solution by Christina Lu

3. **Problem:** Call a cylinder *scalable* if it can be compressed or stretched to be any cylinder with the same volume. Aarush has discovered a scalable cylinder! The current state of Aarush's scalable cylinder has radius 10 and height 30. If the radius of his cylinder when he makes its height double its radius is denoted by r, compute r^3.

   Here is a diagram representing the transformation from $(r,h)=(1,4)$ to $(r,h)=(2,1)$:



   Note: The volume of a cylinder can be computed using πr^2h where r is the radius of the base and h is the height of the cylinder.

   Answer: $\boxed{1500}$

Solution: The volume of Aarush's scalable cylinder is $10^2\pi \cdot 30 = 3000\pi$. When he makes the height double its radius, we can express the volume as $r^2\pi \cdot 2r = 2r^3\pi$. Equating the two expressions, we get that $r^3 = \boxed{1500}$.

Proposed by Tanish Parida, Solution by Christina Lu

4. **Problem:** Anirudh is feeding a sugar syrup solution to the hummingbirds outside his house. He begins with 200 mL of a 6% sugar solution. However, he wants 200 mL of a 15% sugar solution. Given that he has access to an infinite amount of 30% sugar solution, how many mL of the 6% solution must he replace with the 30% solution?

   Answer: $\boxed{75}$

   Solution: His initial solution has $200 \cdot \frac{6}{100} = 12$ mL of sugar and the solution he wants has $200 \cdot \frac{15}{100} = 30$ mL of sugar. Let x be the amount of solution he replaces.

   Then $12 - \frac{6}{100}x + \frac{30}{100}x = 30$. Solving, we get that $x = \boxed{75}$.

   Proposed by Anirudh Pulugurtha, Solution by Christina Lu

5. **Problem:** At a party, there are 50 people, and every possible pair of individuals interact exactly once: every person will high-five everyone they know and shake hands with everyone they don't. If a person A knows person B, then person B also knows person A. If among the 50 people:

   - 10 people know nobody else in the party
   - The remaining 40 people each know exactly 5 other people.

   What is the positive difference between the number of high-fives and handshakes?

   Answer: $\boxed{1025}$

   Solution: Let f be the number of high-fives and s be the number of handshakes. The total number of interactions, $f + s$, is the number of pairs of people, which is $\binom{50}{2} = \frac{50\cdot 49}{2} = 1225$, because for each of the 50 people, they can interact with each of the other 49 people, and then we divide by 2 due to double counting. In particular, if person A knows person B, this connection is counted once for A and once for B. Thus, we have the relation $f + s = 1225$, and so we can first compute the number of high-fives, f.

   To find f, we are given that 10 people know nobody and 40 people each know 5 others, and thus the sum of these is,

   $$10 \cdot 0 + 40 \cdot 5 = 200.$$

   Because this sum is exactly twice the number of high-fives, we find that $f = 200/2 = 100$. Now, we can easily find the number of handshakes,

   $$s = 1225 - f = 1225 - 100 = 1125.$$

   Thus, the answer is $1125 - 100 = \boxed{1025}$.

   Proposed by Nathan Tan, Solution by Aarush Kulkarni

6. **Problem:** If the least common multiple of 6^8, 18^6, 5^{12} and n is 60^{12}, how many distinct values of n are possible?

   Answer: $\boxed{169}$

   Solution: The main idea is to prime factorize the numbers given: $6^8 = 2^8 \cdot 3^8$, $18^6 = 2^6 \cdot 3^{12}$, $5^{12} = 5^{12}$, and $60^{12} = 2^{24} \cdot 3^{12} \cdot 5^{12}$ It thus follows that $\mathrm{lcm}(6^8, 18^6, 5^{12}) = 2^8 \cdot 3^{12} \cdot 5^{12}$.

Therefore, n must have 2^{24} as a factor but it can have any power of 3 and 5, denoted by 3^x, 5^y, in the range $0 \le x, y \le 12$ (each with 13 solutions), giving an answer of $13 \cdot 13 = \boxed{169}$.

Proposed by Eric Huang, Solution by Eric Huang

7. **Problem:** Let A, B, and C be nonzero digits from 1 to 9. Given that $A \div \underline{BB} = 0.\overline{\underline{CB}}$, find $\underline{ABC}$.

   Note: $\underline{A_1 A_2 \ldots A_n}$ for any n digits represents the n digit number with $A_1, A_2, \cdots$, and A_n as its digits. For example, if $A_1 = 2$ and $A_2 = 3$, $\underline{A_1 A_2}$ is just the number 23.

   Answer: $\boxed{918}$

   Solution: First, we can rewrite $\underline{BB}$ as $10B + B = 11B$. Now, we want to rewrite $0.\overline{\underline{CB}}$. By interpreting the place value of each digit, we can conclude that

   $$0.\overline{\underline{CB}} = \frac{C}{10} + \frac{B}{10^2} + \frac{C}{10^3} + \frac{B}{10^4} \cdots .$$

   Now, we can place all the terms with C together and all the terms with B together:

   $$0.\overline{\underline{CB}} = \left(\frac{C}{10} + \frac{C}{10^3} + \cdots\right) + \left(\frac{B}{10^2} + \frac{B}{10^4} + \cdots\right).$$

   These are both infinite geometric series, which we can rewrite as

   $$\left(\frac{C}{10} + \frac{C}{10^3} + \cdots\right) + \left(\frac{B}{10^2} + \frac{B}{10^4} + \cdots\right) = \frac{\frac{C}{10}}{1 - \frac{1}{100}} + \frac{\frac{B}{100}}{1 - \frac{1}{100}} = \frac{10C + B}{99}.$$

   Now, our equation is

   $$\frac{A}{11B} = \frac{10C + B}{99}.$$

   Since A, B, C are all integers, we can conclude that the right side fraction should reduce to the left side. Therefore, $11B$ must be a divisor of 99. This means that B is either 1, 3, or 9.

   Clearly, B cannot be 9, since otherwise $A = 10C + B$, which is impossible since A is one digit while $10C + B$ is two digit.

   If $B = 3$, Then we have $\frac{A}{33} = \frac{10C+3}{99} \Rightarrow 3A = 10C + 3$. So, C must be a multiple of 3 and the minimum value of C is 3. However, this results in $3A = 33 \Rightarrow A = 11$. This is impossible, so B cannot be 3.

   This means that we must have $B = 1$. In this case, we get $9A = 10C + 1$. Now, we can simply plug in values for C until we find that $C = 8, A = 9$ works. Thus, we have $(A, B, C) = (9, 1, 8) \Rightarrow \underline{ABC} = \boxed{918}$.

Proposed by Iris Shi and Tanish Parida, Solution by Nathan Tan

8. **Problem:** Let acute $\triangle ABC$ be inscribed in a circle O. Let chord XY of $\odot O$ be parallel to and equal in length to BC and intersects AB at M and AC at N such that $XM : MA : NY = 1 : 4 : 2$. Given that the ratio of the area of quadrilateral $XYCB$ to $\triangle AMN$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime integers, find $a + b$.

   Answer: $\boxed{5}$

   Solution: Let the lengths be scaled such that $XM = 1$, which implies that $MA = 4$, and $NY = 2$. Consider the following diagram:

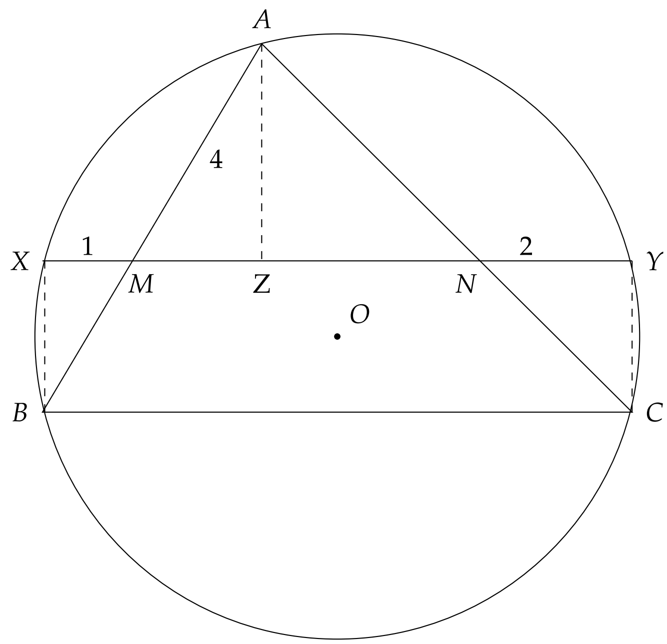


Diagram for Problem 8. Note that the diagram is not drawn to scale.

Set $x := MN$. Then, $BC = XY = x+3$, $MY = x+2$, and $XN = x+1$. By Power of a Point on M with respect to $\odot O$, $XM \cdot MY = AM \cdot MB$, so $1 \cdot (x+2) = 4 \cdot MB \rightarrow MB = \frac{x+2}{4}$. Thus, $AB = MB + 4 = \frac{x+18}{4}$.

Since $XY \;\; BC$, $\triangle AMN \sim \triangle ABC$. Hence, $\frac{AM}{MN} = \frac{AB}{BC}$. This gives us $\frac{4}{x} = \frac{\frac{x+18}{4}}{x+3} \rightarrow x = 6$. Thus, $MB = 2$, and $XY = 9$.

XY and BC are of equal length and parallel so $XYCB$ is a parallelogram. Since opposite angles in an cyclic quadrilateral sum to 180° and opposite angles in a parallelogram are congruent, each of these angles must measure 90°.

Since $\angle MXB = 90^\circ$, $XM = 1$, and $MB = 2$, $\triangle XMB$ is a $30-60-90$. Thus, $XB = \sqrt{3}$ so

$$[XYCB] = XY \cdot XB = 9\sqrt{3}$$

Drop a perpendicular from A to XY and let the intersection be Z. $\angle AZM = 90^\circ$ and

$$\angle AMZ = \angle XMB = 60^\circ$$

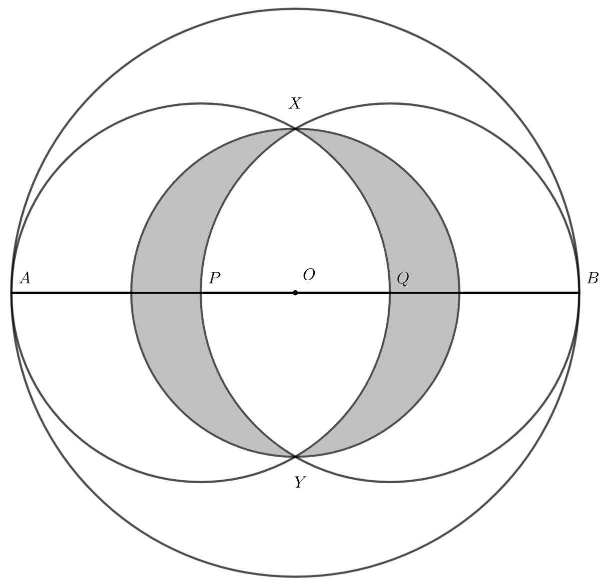
so $\triangle AMZ$ is also a $30-60-90$. We know that $AM = 4$, so $AZ = 2\sqrt{3}$ and

$$[\triangle AMN] = \frac{MN \cdot AZ}{2} = \frac{6 \cdot 2\sqrt{3}}{2} = 6\sqrt{3}$$

Hence, $\frac{[XYCB]}{[\triangle AMN]} = \frac{9\sqrt{3}}{6\sqrt{3}} = \frac{3}{2}$, so our answer is $2+3 = \boxed{5}$.

Proposed by Eric Li, Solution by Christina Lu

9. **Problem:** Diameter $\overline{AB}$ of $\odot O_1$ is trisected by points P and Q such that $AP = PQ = QB = 2$. Circles centered at P and Q are drawn internally tangent to $\odot O$. Another circle $\odot O_2$ drawn centered at O and containing points X and Y is drawn. If the shaded area below can be written as $\frac{a}{b}\pi + c\sqrt{d}$, where a, b, c, d are integers, a and b are relatively prime, and $c\sqrt{d}$ is fully simplified, find $a+b+c+d$.



Answer: $\boxed{9}$

Solution: Consider the following diagram:

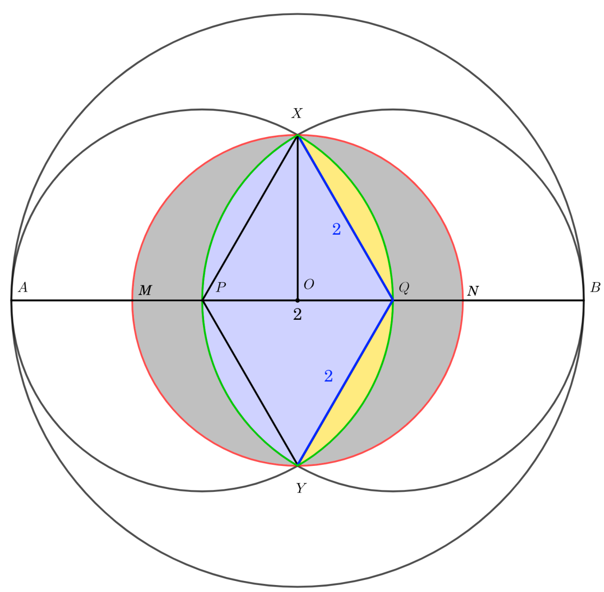


Diagram for Problem 9. Note that the diagram is not drawn to scale

The shaded region is the difference between $\odot O_2$ and the green region. $\odot P$ and $\odot Q$ are both of radius 2, so $PQ = PX = 2$ and $PQ = QX = 2$. Therefore $m\angle PQX = 60^\circ$ and $OX = \sqrt{3}$ by 30-60-90 triangle relations, so the area of $\odot O_2$ is 3π. The green region is composed of the blue circular sector QXY and yellow circular segments QX and QY.

The area of the blue sector QXY is simply $\pi r^2 \cdot \frac{\theta}{360^\circ} = 4\pi \cdot \frac{120^\circ}{360^\circ} = \frac{4}{3}\pi$.

Note that the diagram is symmetric about the line $\overleftrightarrow{XY}$. Thus the yellow area is equal to the area of sector QXY minus the areas of $\triangle PXQ$ and $\triangle PYQ$. We already found the sector's area to be $\frac{4}{3}\pi$, and the triangles are both equilateral triangles with side length 2. Subtracting, we find that

$$\text{Area}_{\text{yellow}} = \text{Area}_{\text{blue}} - (PXQ + PYQ) = \frac{4}{3}\pi - 2\sqrt{3}.$$

Therefore,

$$\text{Area}_{\text{gray}} = \text{Area}_{\text{red}} - \text{Area}_{\text{blue}} - \text{Area}_{\text{yellow}} = 3\pi - \frac{4}{3}\pi - \frac{4}{3}\pi + 2\sqrt{3} = \frac{1}{3}\pi + 2\sqrt{3},$$

so $a = 1, b = 3, c = 2, d = 3$ and $a + b + c + d = \boxed{9}$.

Proposed by Jonathan Ren, Solution by Jonathan Ren

10. **Problem:** Let the sequence $\{a_n\}_{n\in\mathbb{N}}$ be defined by the rule $a_n := n^3 - n + 12$. For each $n \in \mathbb{N}$, we let $d_n = \gcd(a_n, a_{n+1})$. Find the maximum value of d_n.

Answer: $\boxed{36}$

Solution: First, because $d_n = \gcd(a_n, a_{n+1})$, d_n must divide any integer linear combination of a_n and a_{n+1}. Using the Euclidean Algorithm:

$$d_n \mid a_{n+1} - a_n = ((n+1)^3 - (n+1) + 12) - (n^3 - n + 12)$$
$$d_n \mid a_{n+1} - a_n = 3n(n+1).$$

Also notice that $a_n \equiv 12 \pmod{n}$ and $a_n \equiv 12 \pmod{n+1}$. This means that any prime p dividing d_n and n or $n+1$ must divide 12. Thus, the only prime factors of d_n are 2 and 3.

We will now show that the maximum value of d_n is 36.

If we assume that $8 \mid d_n$, then $8 \mid 3n(n+1)$, which means that one of n or $n+1$ is divisible by 8. However, this gives $d_n \equiv 12 \equiv 4 \pmod{8}$, and this contradicts our assumption,

Similarly, if we assume that $27 \mid d_n$, then $27 \mid 3n(n+1)$, so $9 \mid n(n+1)$. This means that one of n or $n+1$ is divisible by 9. However, this gives $d_n \equiv 12 \equiv 3 \pmod{9}$, again contradicting our assumption.

Since $8 \nmid d_n$ and $27 \nmid d_n$, the maximum value of d_n is $2^2 \cdot 3^2 = \boxed{36}$.

Remark: Indeed, we can show that the value 36 is achievable for $n = 3$.

Proposed by Aarush Kulkarni, Solution by Aarush Kulkarni and Eric S Chen

11. **Problem:** Suppose there are one hundred closed doors, labeled consecutively from 1 to 100. One hundred people pass by the doors in sequence. For each $i \in \{1, \ldots, 100\}$, the i-th person selects a door uniformly at random from the set of doors with labels in $\{1, \ldots, i\}$ and opens it. If a door that is already open is selected, it remains open. If the expected number of open doors is ϵ, what is 2ϵ?

Answer: $\boxed{101}$

Solution: The simplest way to think about this is to find the average number of new doors that each person opens.

Case 1: $i = 1$. Door 1 is closed, so they will open 1 new door.

Case 2: i 1. They choose a door from the set $\{1, \ldots, i\}$. We can figure out their chance of opening a new door by calculating the average number of closed doors they have to choose from. A specific door k (where $k \leq i$) is only open if someone from the group $\{k, k+1, \ldots, i-1\}$ has already picked it. The probability that door k remains closed is the product of the probabilities that none of these people pick it, which simplifies neatly to $\frac{k-1}{i-1}$.

Summing this over all possible doors k from 1 to i gives the average number of closed doors available to person i, which is $\sum_{k=1}^{i} \frac{k-1}{i-1} = \frac{i}{2}$. Since person i picks one of the i doors at random, their chance of hitting one of these average $i/2$ closed doors is simply $\frac{i/2}{i} = \frac{1}{2}$, and thus we have $\epsilon = 1 + 99 \times \frac{1}{2} = 50.5$, and $2e = \boxed{101}$.

Proposed by Eric Li and Aarush Kulkarni, Solution by Aarush Kulkarni

12. **Problem:** Consider all the points on a 3-dimensional space with positive integer coordinates. Nathan takes a (not necessarily unique) set of 2026 points with the smallest *Manhattan distances* from the origin. What is the maximum possible Manhattan distance from the origin of one of these points? **Note:** The *Manhattan distance* of two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is equal to $|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$.

    Answer: $\boxed{25}$

    Solution: First, note that for each positive integer triple (x, y, z), its Manhattan distance from the origin is $x + y + z$. Let a_n denote the number of such triples where $x + y + z = n$. The answer to this problem is the smallest integer m such that $a_3 + a_4 + \cdots + a_m = \sum_{n=3}^{m} a_n \geq 2026$.

    Using stars and bars, we know that $a_n = \binom{n-1}{2}$, where we have n stars, 2 bars, and each section must have at least one star. Thus,
$$\sum_{n=3}^{m} a_n = \sum_{n=3}^{m} \binom{n-1}{2} = \sum_{n=2}^{m} \binom{n}{2},$$
    By the hockey stick identity, this simplifies to $\binom{m}{3}$.

    Next, write
$$\binom{m}{3} = \frac{m!}{(m-3)! \cdot 3!} = \frac{m(m-1)(m-2)}{6}$$

    From here, we can easily find that the smallest integer m such that $\frac{m(m-1)(m-2)}{6} \geq 2026$ is $\boxed{25}$.

    Note: The hockey stick identity states that $\sum_{i=a_1}^{a_2} \binom{i}{a_1} = \binom{a_2+1}{a_1+1}$ for integers $a_2 \ a_1 \geq 0$.

Proposed by Nathan Tan, Solution by Christina Lu

13. **Problem:** In triangle ABC, $\angle ABC = 120^\circ$ is bisected by BD. BD is extended to hit the circumcircle of $\triangle ABC$ at point E. Given $AB = 36$ and $BC = 60$, the length of DE can be expressed in the form $\frac{a}{b}$ for some relativity prime integers a, b. Compute $a + b$.

    Answer: $\boxed{149}$

    Solution: Consider the following diagram,

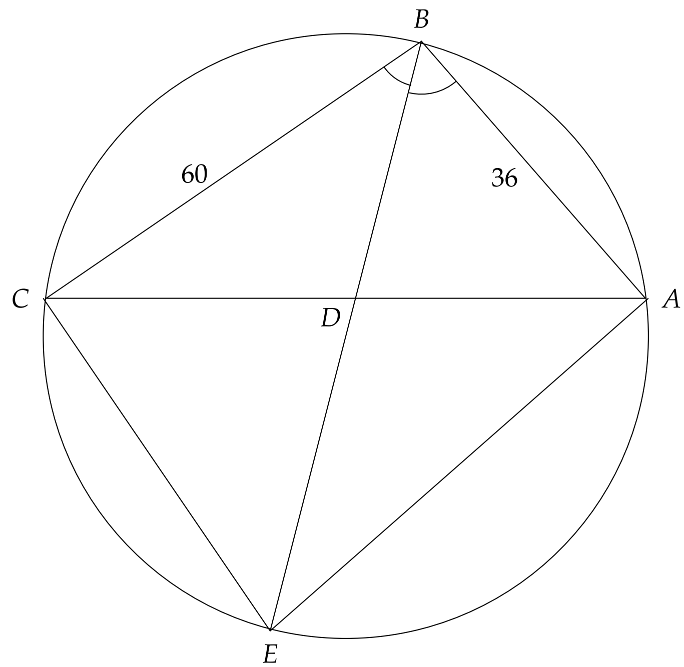


Diagram for Problem 13. Note that the diagram is not drawn to scale

Since $\overline{BE}$ bisects the 120° angle at B, we have $\angle ABE = \angle CBE = 60^\circ$. These inscribed angles subtend arcs $\widehat{AE}$ and $\widehat{CE}$ respectively, so both arcs must measure $2 \cdot 60^\circ = 120^\circ$. Consequently, the remaining arc $\widehat{AC}$ must measure $360^\circ - 120^\circ - 120^\circ = 120^\circ$. As the three arcs $\widehat{AC}$, $\widehat{CE}$, and $\widehat{AE}$ are equal, their corresponding chords are also equal, establishing that $\triangle ACE$ is an equilateral triangle. Let its side length be $s := AC = CE = AE$.

By the Angle Bisector Theorem in $\triangle ABC$, the point D divides AC such that $\frac{AD}{DC} = \frac{AB}{BC} = \frac{36}{60} = \frac{3}{5}$. Since $AD + DC = s$, we find $AD = \frac{3}{8}s$ and $DC = \frac{5}{8}s$. Applying the Power of a Point theorem to point D with respect to the circumcircle gives $AD \cdot DC = BD \cdot DE$, so $BD \cdot DE = \frac{3}{8}s \cdot \frac{5}{8}s = \frac{15}{64}s^2$.

Let $n := BD$ and $p := DE$. Applying Stewart's Theorem to $\triangle ACE$ with cevian ED from vertex E to side AC, we have $AC(ED^2 + AD \cdot DC) = CE^2 \cdot AD + AE^2 \cdot DC$. Substituting our values gives $s(p^2 + \frac{15}{64}s^2) = s^2(\frac{3}{8}s) + s^2(\frac{5}{8}s) = s^3$, which simplifies to $p^2 = \frac{49}{64}s^2$, meaning $p = \frac{7}{8}s$. Now using the Power of a Point result, $np = \frac{15}{64}s^2$, we can find n: $n(\frac{7}{8}s) = \frac{15}{64}s^2$, which simplifies to $n = \frac{15}{56}s$.

Finally, we apply Stewart's Theorem to $\triangle ABC$ with cevian BD. This gives $AC(BD^2 + AD \cdot DC) = AB^2 \cdot DC + BC^2 \cdot AD$. Substituting known values yields $s(n^2 + \frac{15}{64}s^2) = 36^2(\frac{5}{8}s) + 60^2(\frac{3}{8}s) = 810s + 1350s = 2160s$. This simplifies to

$$n^2 + \frac{15}{64}s^2 = 2160.$$

We can now solve for n by substituting $s = \frac{56}{15}n$ into this equation:

$$n^2 + \frac{15}{64}\left(\frac{56}{15}n\right)^2 = 2160.$$

This becomes $n^2 + \frac{49}{15}n^2 = 2160$, so $\frac{64}{15}n^2 = 2160$, which gives $n^2 = 2160 \cdot \frac{15}{64} = \frac{2025}{4}$, and thus $n = \frac{45}{2}$. The desired length is $DE = p$, and we know $p = \frac{7}{8}s = \frac{7}{8}(\frac{56}{15}n) = \frac{49}{15}n$. Substituting the value of n, we find

$$p = \frac{49}{15} \cdot \frac{45}{2} = \frac{147}{2}.$$

The length of DE is $\frac{147}{2}$, so $a = 147$ and $b = 2$. These are relatively prime, and their sum is $a + b = \boxed{149}$.

Alternate Solution: Uses Trigonometry. First, we compute the length of side AC using the Law of Cosines in $\triangle ABC$:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(120^\circ) \\ &= 36^2 + 60^2 - 2(36)(60)\left(-\frac{1}{2}\right) \\ &= 1296 + 3600 + 2160 \\ &= 7056. \end{aligned}$$

Taking the square root, we find $AC = 84$. By the Angle Bisector Theorem, point D divides AC such that $\frac{AD}{DC} = \frac{AB}{BC} = \frac{36}{60} = \frac{3}{5}$. Since $AD + DC = AC = 84$, we find $AD = \frac{3}{8}(84) = 31.5$ and $DC = \frac{5}{8}(84) = 52.5$.

Next, we find the length of the angle bisector BD using the formula $BD^2 = AB \cdot BC - AD \cdot DC$,

$$BD^2 = (36)(60) - (31.5)(52.5) = 2160 - 1653.75 = 506.25.$$

Thus $BD = \sqrt{506.25} = 22.5$. Finally, applying the Power of a Point theorem to point D with respect to the circumcircle of $\triangle ABC$ gives the relation $AD \cdot DC = BD \cdot DE$, from which we can solve for DE:

$$DE = \frac{AD \cdot DC}{BD} = \frac{(31.5)(52.5)}{22.5} = \frac{1653.75}{22.5} = 73.5 = \frac{147}{2}.$$

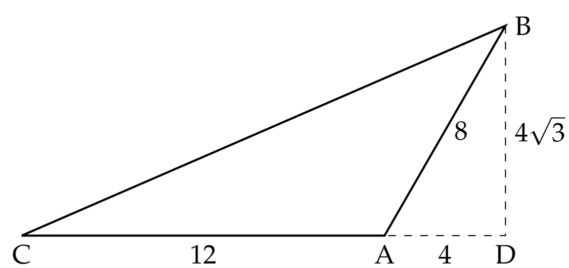
The length of DE is $\frac{147}{2}$, so $a = 147$ and $b = 2$, thus desired sum is $a + b = 147 + 2 = \boxed{149}$.

Proposed by Steven Feng, Solution by Anirudh Pulugurtha, Eric S Chen, and Aarush Kulkarni

14. **Problem:** In triangle ABC, $\angle BAC = 120^\circ$ and is quadrisected (cut into 4 equal angles). The quadrisectors intersect BC at X, Y, and Z such that B, X, Y, Z, and C lie on a line in that order. Given $AB = 8$ and $AC = 12$, find $a + b$ where $(AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC) = a\sqrt{b}$.

Answer: $\boxed{99}$

Solution: We will use area manipulations to solve this problem. First, let us find the area of $\triangle ABC$.



After dropping the altitude from B onto AC, notice that $\triangle ABD$ is a 30-60-90 triangle. Thus, $[\triangle ABC] = \frac{1}{2} \cdot 12 \cdot 4\sqrt{3} = 24\sqrt{3}$.

Next, note that the quadrisectors AX, AY, and AZ divide the 120° angle at A into four equal angles of 30° each. These segments partition $\triangle ABC$ into four smaller triangles: $\triangle ABX, \triangle AXY, \triangle AYZ$, and $\triangle AZC$, where the sum of the areas of these four triangles equals the area of $\triangle ABC$. For a triangle with sides u and v enclosing a 30° angle, its area is $\frac{1}{4}uv$. This is because the altitude to the side of length v is $u/2$; applying this, we find the areas of the smaller triangles,

$$[\triangle ABX] = \frac{1}{4}(AB)(AX), [\triangle AXY] = \frac{1}{4}(AX)(AY)$$

$$[\triangle AYZ] = \frac{1}{4}(AY)(AZ), [\triangle AZC] = \frac{1}{4}(AZ)(AC).$$

Summing these areas gives the total area of $\triangle ABC$:

$$[\triangle ABC) = \frac{1}{4}\left((AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC)\right).$$

We can now substitute the known area of $\triangle ABC$ to find the value of the expression,

$$24\sqrt{3} = \frac{1}{4}\left((AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC)\right).$$

Multiplying both sides by 4 gives the final result,

$$(AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC) = 4 \cdot 24\sqrt{3} = 96\sqrt{3},$$

and thus the desired sum is $a + b = 96 + 3 = \boxed{99}$.

Alternate Solution: Uses Trigonometry. The sine area formula states that the area of any given triangle is $\frac{1}{2}ab\sin C$ where a and b are two side lengths of the triangle and C is the measure of the angle between the two sides. We can use this to compute the area of $\triangle ABC$ as follows,

$$[ABC] = \frac{1}{2}(8)(12)\sin 120^\circ = 24\sqrt{3}.$$

We can also compute the area of $\triangle ABC$ by summing the areas of $\triangle ABX$, $\triangle AXY$, $\triangle AYZ$, and $\triangle AZC$. By the sine area formula, the areas of those 4 triangles are:

$$[ABX] = \frac{1}{2}(AB)(AX)\sin 30^\circ, [AXY] = \frac{1}{2}(AX)(AY)\sin 30^\circ,$$

$$[AYZ] = \frac{1}{2}(AY)(AZ)\sin 30^\circ, [AZC] = \frac{1}{2}(AZ)(AC)\sin 30^\circ,$$

and summing over these areas yield,

$$\begin{aligned}[ABC] = 24\sqrt{3} &= [ABX] + [AXY] + [AYZ] + [AZC] \\ &= \frac{1}{2}\sin 30^\circ[(AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC)].\end{aligned}$$

Thus, $(AB)(AX) + (AX)(AY) + (AY)(AZ) + (AZ)(AC) = 24\sqrt{3} \cdot \frac{2}{\sin 30} = 96\sqrt{3}$, and $a + b = 96 + 3 = \boxed{99}$.

Remark: The first "area manipulation" solution is actually a geometric special case of the trigonometric method where the included angle is 30°, making the sine factor $\frac{1}{2}$.

Proposed by Steven Feng, Solution by Anirudh Pulugurtha, Eric S Chen, and Aarush Kulkarni

15. **Problem:** Ben rolls a fair six-sided die 2025 times. If the probability that the the sum of all results he gets is divisible by 7 is $\frac{m}{6^{2025}}$, then find the last 5 digits of m.

Answer: $\boxed{65910}$

Solution: The main idea is to find a recurrence relation for the probabilities and solve for m.

Let $p_n(k)$ be the probability that the sum of results after n rolls is congruent to $k \pmod 7$. To obtain a sum congruent to $k \pmod 7$ after n rolls, the sum after $n-1$ rolls must have been congruent to $k - j$

(mod 7), where $j \in \{1, \ldots, 6\}$ is the outcome of the n-th roll. Since each outcome j has a probability of $1/6$, we have the recurrence

$$p_n(k) = \frac{1}{6} \sum_{j=1}^{6} p_{n-1}(k - j \pmod 7).$$

For our case $k = 0$, the previous sum must have been congruent to one of $1, 2, \ldots, 6 \pmod 7$. Thus,

$$p_n(0) = \frac{1}{6} \sum_{j=1}^{6} p_{n-1}(j).$$

Because $\sum_{i=0}^{6} p_{n-1}(i) = 1$, we can simplify this to $p_n(0) = \frac{1}{6}(1 - p_{n-1}(0))$. Let $a_n := p_n(0)$. The recurrence is $a_n = \frac{1}{6}(1 - a_{n-1})$ with the initial condition $a_0 = 1$, as the sum after zero rolls is 0. The fixed point of this recurrence is $1/7$. We find the general solution is $a_n = \frac{1}{7} + C(-\frac{1}{6})^n$. Using $a_0 = 1$, we find $1 = 1/7 + C$, which implies $C = 6/7$. Therefore,

$$a_n = \frac{1}{7} + \frac{6}{7}\left(-\frac{1}{6}\right)^n.$$

We need to find a_{2025}. Since 2025 is odd, we obtain

$$a_{2025} = \frac{1}{7} - \frac{6}{7 \cdot 6^{2025}} = \frac{6^{2024} - 1}{7 \cdot 6^{2024}}.$$

The problem states this probability is $\frac{m}{6^{2025}}$, so we set these equal,

$$\frac{m}{6^{2025}} = \frac{6^{2024} - 1}{7 \cdot 6^{2024}} = \frac{6(6^{2024} - 1)}{7 \cdot 6^{2025}} = \frac{6^{2025} - 6}{7 \cdot 6^{2025}}.$$

This implies that $m = \frac{6^{2025} - 6}{7}$. To find the last five digits of m, we compute $X := 7m = 6^{2025} - 6$ modulo 10^5. Note that we can use the Chinese Remainder Theorem, since $10^5 = 32 \cdot 3125$.

First, for modulo 32, since $2025 \geq 5$, 6^{2025} is divisible by $2^5 = 32$. This implies $X \equiv 0 - 6 = -6 \equiv 26 \pmod{32}$.

Next, for modulo $3125 = 5^5$, we use the binomial expansion:

$$\begin{aligned} 6^{2025} = (1+5)^{2025} &= \sum_{i=0}^{2025} \binom{2025}{i} 5^i \\ &\equiv \binom{2025}{0} 5^0 + \binom{2025}{1} 5^1 + \binom{2025}{2} 5^2 \pmod{3125}. \end{aligned}$$

For the sake of clarity, the terms for $i \geq 3$ are congruent to $0 \pmod{3125}$ because their 5-adic valuation is at least $v_5(\binom{2025}{3} 5^3) = v_5(2025) + v_5(125) = 2 + 3 = 5$.

We can compute the remaining terms by hand as follows, $\binom{2025}{0} 5^0 = 1$; $\binom{2025}{1} 5^1 = 10125 \equiv 750 \pmod{3125}$; and $\binom{2025}{2} 5^2 = \frac{2025 \cdot 2024}{2} \cdot 25 \equiv 1250 \pmod{3125}$. Summing these gives $6^{2025} \equiv 1 + 750 + 1250 = 2001 \pmod{3125}$, so $X \equiv 2001 - 6 = 1995 \pmod{3125}$.

We now solve the system of congruences $X \equiv 26 \pmod{32}$ and $X \equiv 1995 \pmod{3125}$. From the second congruence, $X = 3125k + 1995$. Substituting into the first gives

$$\begin{aligned} 3125k + 1995 &\equiv 26 \pmod{32} \\ 21k + 11 &\equiv 26 \pmod{32} \\ 21k &\equiv 15 \pmod{32} \\ 11k &\equiv 17 \pmod{32}. \end{aligned}$$

The inverse of 11 $\pmod{32}$ is 3. Multiplying by 3 gives $k \equiv 17 \cdot 3 = 51 \equiv 19 \pmod{32}$. So $k = 32j + 19$ for some integer j. Substituting this back gives

$$X = 3125(32j + 19) + 1995 = 100000j + 59375 + 1995 = 100000j + 61370.$$

Thus, $7m \equiv 61370 \pmod{10^5}$. This is equivalent to $7m = 100000k' + 61370$ for some integer k'. For the right hand side to be divisible by 7, we need $100000k' + 61370 \equiv 5k' + 1 \equiv 0 \pmod{7}$, which implies $5k' \equiv 6 \pmod{7}$. Multiplying by 3 (the inverse of 5) gives $k' \equiv 18 \equiv 4 \pmod{7}$. The smallest non negative value for k' is 4. We substitute $k' = 4$ to find m:

$$m = \frac{100000(4) + 61370}{7} = \frac{461370}{7} = \boxed{65910}.$$

Remark: The expression for m can also be derived using generating functions. The number of ways to obtain a sum divisible by 7 after $N = 2025$ rolls is given by the root of unity filter,

$$m = \frac{1}{7}\sum_{k=0}^{6} P(\omega^k)^N,$$

where $P(x) = x + \cdots + x^6$ and $\omega = e^{2\pi i/7}$. We have $P(1) = 6$, and for $k \in \{1, \ldots, 6\}$, $P(\omega^k) = -1$. Since $N = 2025$ is odd, this gives

$$m = \frac{1}{7}(6^{2025} + 6(-1)^{2025}) = \frac{6^{2025} - 6}{7}.$$

Proposed by Eric Huang, Solution by Aarush Kulkarni