Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 19 — Sunday, October 20, 2024

1. **Problem:** 20 years ago, my father was 4 times my age. Today, he is twice my age. How old am I today?

   Solution: Let the father's age today be x, and his age 20 years ago be $x - 20$. Similarly, let the child age today be y, and their age 20 years ago be $y - 20$. Using these newly defined variables, we can write the following systems of equations:

$$x - 20 = 4(y - 20) \quad (1)$$
$$x = 2y \quad (2)$$

   Plugging in (2) into (1), we get

$$2y - 20 = 4(y - 20)$$

   Solving this equation leads us to the answer $y = \boxed{30}$.

   Proposed by Ben Zhu

2. **Problem:** Omar rolls 2 dice. Let $\frac{a}{b}$ be the probability that the sum of the two numbers is a prime number, where a and b are relatively prime. Find $a + b$.

   Solution: When rolling two dice, the number of different ways to roll it is equal to $6 \cdot 6$ which is equal to 36. There are a total of five prime numbers that can be obtained, and those are 2, 3, 5, 7 and 11. There is 1 possible way to get two, 2 possible ways to get three, 4 possible ways to get five, 6 possible ways to get seven, and only 2 possible ways to get eleven. After adding them all, the probability is $15/36 = 5/12 \Rightarrow \boxed{17}$.

   Proposed by Omar Graia

3. **Problem:** Hillary has a disk of radius 6 and one sphere of radius n. If the value of the surface area of the sphere is equal to the area of the disk, then find the value n.

   Solution: The surface area of a sphere is $4\pi n^2$ where n is the radius of the sphere. The area of a circle is πr^2. We can solve for n by plugging in $r = 6$ to get

$$4\pi \cdot n^2 = \pi \cdot 6^2 \Rightarrow n = \boxed{3}.$$

   Proposed by Raymond Gao

4. **Problem:** Find the sum of all positive integers n such that n is a divisor of $n^2 + 5$.

   Solution: We know that the following value is an integer,

$$\frac{n^2 + 5}{n} = n + \frac{5}{n}.$$

   Therefore, we need $5/n$ to be an integer, where n is a positive integer. This only happens when $n = 1$ and $n = 5$, so the answer is $1 + 5 = \boxed{6}$.

   Proposed by Ayaan Garg

5. **Problem:** Aiden is running on a soccer field and he suddenly passes out! Luckily, he is in the middle of the field in the center circle such that anyone can run over him. He lies perpendicular to the middle line such that his toes touch the middle line and his head touches the circle. Given that the circle has a diameter of 60 feet, Aiden is 6 feet tall, and the distance between his toes to the center of the field can be written as $\sqrt{b}$ feet, find b.

Solution: To solve this problem, we can use the Pythagorean theorem to find the distance between Aidan's toes to the center of the field. Since the radius of the circle is 30 and Aidan is 6 feet tall, the Pythagorean formula can be written as $6^2 + x^2 = 30^2$, where x is the distance of Aidan's toes from the center of the field. Solving this, we get $\sqrt{900-36}$ or $\sqrt{864}$. The desired quantity is $\boxed{864}$.

Proposed by Christopher Zhang

6. **Problem:** A *funky* date is a date where, when written in the form MM/DD/YYYY, all of the nonzero digits are the same. Today is February 2nd, 2022. Find the number of days between the *funky* dates directly before and directly after this date.

   Solution: We know that on the funky date directly before 02/02/2022, the nonzero digit is a 2 because of the year. Since no funky date before 02/02/2022 is in 2022, we know it must be in 2020. Hence, the last date that satisfies the given requirements is 02/22/2020. We also know that the date directly after 02/02/2022 would be 02/20/2022. Since we exclude the endpoint dates, the distance between these days is $366 + 365 - 2 - 1 = \boxed{728}$.

   Note: There is a leap day in 2020.

   Proposed by Raymond Gao

7. **Problem:** How many pairs of positive integers (a, b) for $a < b$ satisfy $\frac{1}{a} + \frac{3}{b} = \frac{1}{2024}$?

   Solution: We can multiply both sides of the equation $2024ab$ and get

$$\begin{aligned} 2024b + 6072a &= ab \\ ab - 6072a - 2024b &= 0 \\ (a - 2024)(b - 6072) &= 6072 \cdot 2024 \\ &= 2^6 \cdot 3 \cdot 11^2 \cdot 23^2 \end{aligned}$$

   Because there is a solution when $a = b = 8096$, we are looking for all solutions where a is less than 8096. This is equivalent to all positive factors of $6072 \cdot 2024$ that are less than 6072. This is equivalent to the total number of factors minus the number of factors that do not exceed 2024.

   We can count the number of factors that do not exceed 2024 in two different cases:

   Case I (no factor of 3): In this case, we are looking for factors of $2^6 \cdot 11^2 \cdot 23^2$ that do not exceed 2024. Interestingly, 2024 is the square root of that number. The total number of factors of $2^6 \cdot 11^2 \cdot 23^2$ is $7 \cdot 3 \cdot 3 = 63$. The number of working factors is half of one more than that, or 32.

   Case II (factor of 3): In this case, we are looking for factors of $2^6 \cdot 3 \cdot 11^2 \cdot 23^2$ that must include a factor of 3 that do not exceed 2024. Because we know that there is a factor of 3, this is equivalent to looking for factors of $2^6 \cdot 11^2 \cdot 23^2$ that are less than $\frac{2024}{3}$, or less than 675. This can be found pretty easily by casework on the power of 11 and 23, finding the possible powers of 2. If one of them is squared, the other one can not be present.

   - $(11^0, 23^0)$ means that any of the 7 powers of 2 are possible
   - $(11^1, 23^0)$ means that 64 is not possible, but the other 6 work
   - $(11^0, 23^1)$ means that 32 and 64 are not possible, but the other 5 work
   - $(11^1, 23^1)$ means that only 1 and 2 work giving 2 possibilities
   - $(11^2, 23^0)$ means that 1, 2 and 4 work giving 3 possibilities
   - $(11^0, 23^2)$ means that only the number 1 works.

   The number of working cases is $7 + 6 + 5 + 2 + 3 + 1 = 24$.

   Adding the two larger cases, we get that there are 56 factors that do not exceed 2024. There are 126 $7 \cdot 2 \cdot 3 \cdot 3$ factors of the large number, so there are $\boxed{70}$ factors less than 6072, and subsequently 70 positive integer pairs that work in the original equality.

Proposed by Iris Shi

8. **Problem:** For a triplet of positive integers (a, b, c); we have $\gcd(a, b) = 12$, $\gcd(b, c) = 12$, $\gcd(c, a) = 120$; and $\operatorname{lcm}(a, b) = 1680$, $\operatorname{lcm}(b, c) = 2520$, $\operatorname{lcm}(c, a) = 720$. Find $a + b + c$.

   Solution: We can use the fact that, for positive integers x and y, $\gcd(x, y) \cdot \operatorname{lcm}(x, y) = x \cdot y$. Thus,

$$a \cdot b = \gcd(a, b) \cdot \operatorname{lcm}(a, b) = 12 \cdot 1680 = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$$

$$b \cdot c = \gcd(b, c) \cdot \operatorname{lcm}(b, c) = 12 \cdot 2520 = 30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$$

$$c \cdot a = \gcd(c, a) \cdot \operatorname{lcm}(c, a) = 120 \cdot 720 = 86400 = 2^7 \cdot 3^3 \cdot 5^2$$

   Multiplying the equations together, we have

$$a^2 \cdot b^2 \cdot c^2 = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2$$

$$a \cdot b \cdot c = 2^9 \cdot 3^4 \cdot 5^2 \cdot 7$$

   Dividing this by one of the three equations above will give us the variable that is not present in the equation. For example, if we divide abc by ab, we will be left with c. Thus, we get that,

$$a = 2^4 \cdot 3 \cdot 5 = 240$$
$$b = 2^2 \cdot 3 \cdot 7 = 84$$
$$c = 2^3 \cdot 3^2 \cdot 5 = 360.$$

   The desired quantity is $a + b + c = 240 + 84 + 360 = \boxed{684}$.

   Proposed by Daniel Cai

9. **Problem:** A caterpillar wants to go from $(0, 0)$ to $(5, 5)$ on the coordinate plane. It can only move up and right. If it reaches $(3, 3)$ at any point in time, it teleports to $(1, 1)$. Determine how many ways the caterpillar can reach its destination within 20 moves.

   Solution: This problem can be solved using casework. To establish the cases, we find that the maximum number of times the caterpillar can cross (3, 3) and restart is **2** times. This is because it takes 10 moves to reach (5, 5), but starting at (1, 1) leaves 8 moves to reach (5, 5). If the caterpillar touches (3, 3) twice, it will take exactly $6 + 6 + 8 = 20$ moves to get to (5, 5).

   Case I: Path does not cross $(3, 3)$. The caterpillar can only move up and right, so 5 "right" moves and 5 "up" moves are needed to reach the finish. Since these moves can be ordered in any combination, the total number of paths to point (5, 5) is $\binom{10}{5}$ = 252, meaning 5 "up" or "right" moves are being chosen out of the 10 total moves. However, this path cannot cross (3, 3). Following the previous method, the number of paths to (5, 5) that do **not** cross (3, 3) is $252 - \binom{6}{3} \cdot \binom{4}{2} = \mathbf{132}$. $\binom{6}{3}$ is the number of paths *to* point (3, 3), while $\binom{4}{2}$ is the number of paths *from* (3, 3). By multiplying these two numbers, we can find the number of paths that contain points (3, 3).

   Case II: Path crosses $(3, 3)$ only once. In this case, the caterpillar will essentially start twice; the first time will take it to (3, 3) and leave it at (1, 1) and the second time it will avoid (3, 3) to reach its destination. The number of paths to (3, 3) is $\binom{6}{3} = 20$. When it starts again at (1, 1), it now has $\binom{8}{4} = 70$ potential ways to get to (5, 5). Since it can not cross (3, 3) again in this case, we subtract $\binom{4}{2} \cdot \binom{4}{2} = 36$, leaving us with $70 - 36 = 34$ ways to get to our destination from (1, 1). Multiplying these 2 paths together, there is $20 \cdot 34 = \mathbf{680}$ paths the caterpillar can take if it touches (3, 3) only once.

   Case III: Path crosses $(3, 3)$ only twice. We have found that the number of paths to (3, 3) from (0, 0) is 20. After touching (3, 3), the caterpillar is brought to (1, 1). From here, there are $\binom{4}{2} = 6$ ways to get to (3, 3) a second time. Finally, using numbers we found in case II, we have $20 \cdot 6 \cdot 34 = \mathbf{4080}$ ways.

Finally, since the cases are independent, we can add the results from each case to get $132 + 680 + 4080 = \boxed{4892}$ total paths.

Proposed by Christina Lu

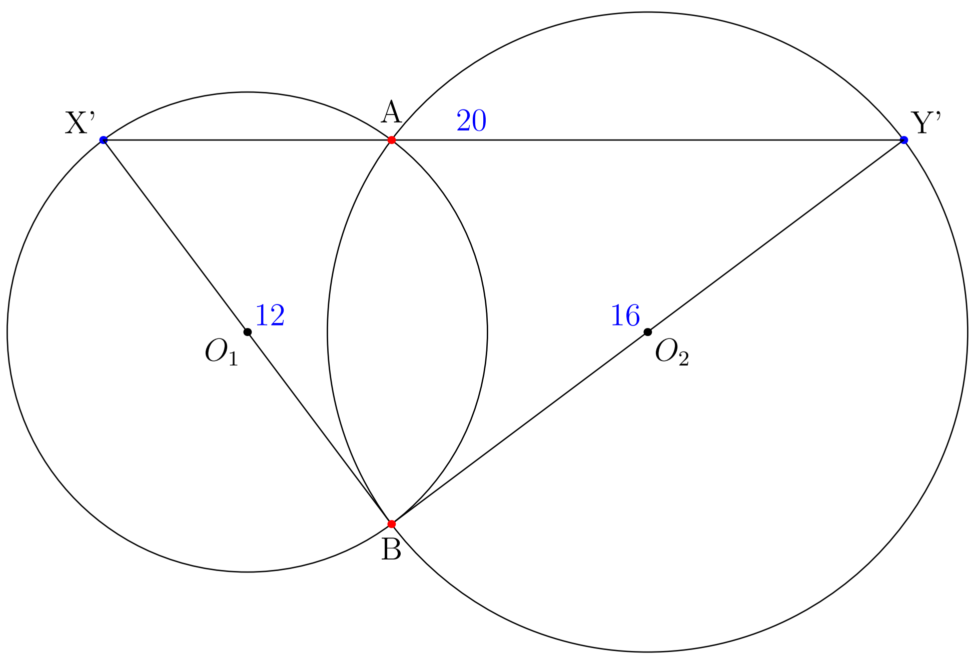
10. **Problem:** Circles ω_1 and ω_2 have radii 6 and 8, respectively, with the distance between their centers being 10. The two circles intersect at points A and B. A line through A intersects the two circles at points X and Y. What is the maximum length of XY?

    Solution: Let the centers of the circles be O_1 and O_2, respectively. Since $O_1B = 6, O_2B = 8$, and $O_1O_2 = 10, O_1B \perp O_2B$.

    Now, draw two arbitrary XY through A and call them X_1Y_1 and X_2Y_2, with X_1 and X_2 on ω_1 and Y_1 and Y_2 on ω_2. Then, draw segments X_1B, X_2B, Y_1B and Y_2B. Since angles X_1Y_1B and X_2Y_2B are inscribed in the same arc, they are equal. Similarly, angles Y_1X_1B and Y_2X_2B are equal. Therefore, $\triangle X_1Y_1B \sim \triangle X_2Y_2B$.

    This implies that all triangles constructed in this manner are similar, so if we maximize the lengths of XB and YB, we can maximize XY. The maximum length of XB is clearly the diameter of circle ω_1, and the maximum length of YB is clearly the diameter of circle ω_2.

    Let X' and Y' be values of X and Y that maximize XY. Then, we have a triangle with side lengths BX', BY', and $X'Y'$. Since BX' and BY' are the diameters of ω_1 and ω_2 respectively, their lengths are 12 and 16 respectively. Additionally, we know that $\angle O_1BO_2 = 90^\circ$ because $O_1B \perp O_2B$. Thus, we can use the Pythagorean Theorem to get $X'Y' = \sqrt{12^2 + 16^2} = \boxed{20}$.



Proposed by Eric Xiang

11. **Problem:** There are four identical spheres, each with a radius of 1, positioned in space such that every sphere touches the other three externally (their surfaces are tangent at exactly one point for each pair). Now, a fifth sphere is placed among them such that it touches all four of these spheres externally as well. The diameter of the fifth sphere can be written as $\sqrt{a} - b$ for positive integers a and b. What is $a + b$?

Solution: Let us begin by understanding the configuration given.

We have four identical spheres, each with a radius of 1, and each sphere is positioned so that every sphere touches the other three externally. This configuration forms the vertices of a *regular tetrahedron* with an edge length double the radius of 1, which implies it has an edge length of 2.

Furthermore, we know that the center of the fifth sphere is located at the center of the tetrahedron. To find the circumradius of a tetrahedron (distance between a vertex of the tetrahedron and the center of the tetrahedron), one can use the following formula,

$$R = \frac{a\sqrt{6}}{4},$$

where a is the edge length of the tetrahedron and R is the circumradius.

Quick Derivation of Formula: We consider a regular tetrahedron $ABCD$ with all edges of equal length a. The centroid G of $\triangle ABC$ is located at the distance $AG = \frac{\sqrt{3}}{3}a$, which is from the property of the centroid dividing the median in a 2:1 ratio. We can then calculate the height, h, of the tetrahedron using the Pythagorean theorem:

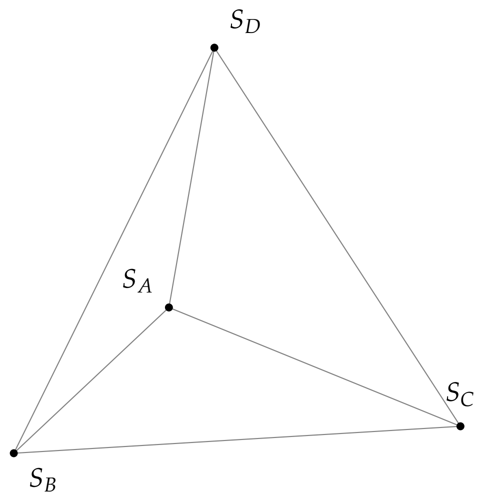
$$h = \sqrt{a^2 - \left(\frac{\sqrt{3}}{3}a\right)^2} = \sqrt{a^2 - \frac{a^2}{3}} = \sqrt{\frac{2a^2}{3}} = \frac{a\sqrt{6}}{3}.$$

Let M be the midpoint of BC. We have $AD = a$, and $AM = DM = \frac{\sqrt{3}}{2}a$. DG is the altitude of ADM if we consider AM to be the base. Now, let the center of $ABCD$ be O, and we know O lies on DG. Let $AO = DO = R$, where R is the circumradius of the tetrahedron. Since $\triangle AOG$ is a right triangle, $DG = \frac{\sqrt{6}}{3}a$, and $OG = \frac{\sqrt{6}}{3}a - R$, we have

$$\begin{gathered} AG^2 + OG^2 = AO^2 \\ \left(\frac{\sqrt{3}}{3}a\right)^2 + \left(\frac{\sqrt{6}}{3}a - R\right)^2 = R^2 \\ \frac{a^2}{3} + \frac{2a^2}{3} - \frac{2aR\sqrt{6}}{3} + R^2 = R^2 \\ a^2 = \frac{2aR\sqrt{6}}{3} \\ R = \frac{a\sqrt{6}}{4} \end{gathered}$$

Plugging in $a = 2$, we find that the circumradius of the tetrahedron is $\frac{\sqrt{6}}{2}$. Since the fifth sphere is externally tangent to the other four spheres, we have that the radius of the fifth sphere is 1 less than the circumradius of the tetrahedron due to the positioning, which implies that the radius of the fifth sphere is $\frac{\sqrt{6}}{2} - 1$. Thus, the diameter is $\sqrt{6} - 2 \Rightarrow \boxed{8}$.

A picture is shown below for better visualization, where S_A, S_B, S_C, and S_D represent the centers of the four original spheres:



Proposed by Eric Li

12. **Problem:** The cubic $x^3 + 8x^2 + 5x + 3$ has roots r, s, and t. There exists a unique set of numbers A, B, and C, such that $x^3 + Ax^2 + Bx + C$ has $r + s$ as a root. What is value of $(A + B + C)^2$?

Solution: Let $P(x) = x^2 + 8x^2 + 5x + 3$. By Vieta's formula on $P(x)$, we have

$$\begin{aligned} r + s + t &= -8/1 = -8 \\ rs + rt + st &= 5/1 = 5 \\ rst &= -3/1 = 3. \end{aligned}$$

The cubic $x^3 + Ax^2 + Bx + C$ has roots $r + s$, $r + t$, and $s + t$ by symmetry. By Vieta's formula on $x^3 + Ax^2 + Bx + C$,

$$\begin{aligned} (r + s) + (r + t) + (s + t) &= -A/1 \\ 2(r + s + t) &= -A \\ 2(-8) &= -A \\ A &= 16. \end{aligned}$$

We notice that $r^2 + s^2 + t^2 = (r + s + t)^2 - 2(rs + rt + st)$. Again, with Vieta's formula:

$$\begin{aligned} (r + s)(r + t) + (r + s)(s + t) + (r + t)(s + t) &= B/1 \\ r^2 + s^2 + t^2 + 3(rs + rt + st) &= B \\ (r + s + t)^2 + rs + rt + st &= B \\ (-8)^2 + 5 &= B \\ B &= 69. \end{aligned}$$

Lastly, we find C. Notice that since $r + s + t = -8$, we have $r + s = -8 - t$. A similar statement is true for $r + t$ and $s + t$.

$$\begin{aligned} (r + s)(r + t)(s + t) &= -C/1 \\ (-8 - t)(-8 - s)(-8 - r) &= -C \end{aligned}$$

Notice that the left-hand side is simply $P(-8)$.

$$\begin{aligned}(-8)^3+8\cdot(-8)^2+5(-8)+3&=C\\ -37&=-C\\ C&=37.\end{aligned}$$

Therefore, $(A+B+C)^2=(16+69+37)^2=122^2=\boxed{14884}$.

Proposed by Ayaan Garg

13. **Problem:** Let $f(x)$ be the sum of the factors of x. Given that b is the smallest positive integer with 6^6 factors, find the largest prime factor of $f(b)$.

Solution: Any number can be written as a product of its factors. For the prime factorization of a number in the form ${p_1}^{a_1}\cdot{p_2}^{a_2}\cdots{p_n}^{a_n}$, we can "build" each factor by choosing to use any amount of each prime factor. Since we can also choose to use 0 of a prime factor, each prime factor has a total of a_i+1 ways to be involved in the factor we are "building". Thus, the total number of factors that the number has can be written as $(a_1+1)\cdot(a_2+1)\cdot\cdots\cdot(a_n+1)$.

For our number b, we can start with a simple number that has 6^6 factors. In this case, we start with $2^5\cdot3^5\cdot5^5\cdot7^5\cdot11^5\cdot13^5$. Now, we will attempt to minimize this number by either adding another prime factor, or changing the exponent of an existing prime factor. As we change the prime factorization, our end goal is to make b as small as possible. The table below shows the prime factors of our current minimum number on the top row and 1 plus the exponents of the prime factors on the bottom. Note that the product of the numbers on the bottom should always equal 6^6. We begin with:

2	3	5	7	11	13
6	6	6	6	6	6

To minimize b, we can begin by dividing the exponent of 13 (the largest prime) and multiplying the exponent of 2 (the smallest prime). Specifically, we can change 13^5 to 13^1 and 2^5 to 2^{17} because $2^{12}<13^4$ Our table is now:

2	3	5	7	11	13
18	6	6	6	6	2

Now, we look at the second smallest prime and the second largest prime. We change 3^5 to 3^{11} and 11^5 to 11^2 as $3^6<11^3$. Note that we can't change 3^5 to 3^{17} and 11^5 to 11^1 as $3^{12}>11^4$. This makes our table

2	3	5	7	11	13
18	12	6	6	3	2

We can't change the exponents of 5 and 7, as changing 5^5 to 5^{11} and 7^5 to 7^2 would make it larger instead ($5^6>7^3$) So, we begin to introduce more prime factors. First, we can add on a 17, changing the 2^{17} down to a 2^8, then adding 19 with 3^{11} decreasing to 3^5. Similarly, we can also add 23 and 29, and change 5^5 and 7^5 to 5^2 and 7^2. Our table is now

2	3	5	7	11	13	17	19	23	29
9	6	3	3	3	2	2	2	2	2

This is the minimum possible number with 6^6 factors, as we cannot add another prime. Adding 31, the next smallest prime, would cause us to have to decrease the 3^5 to 3^2 at most, and $3^3 = 27 < 31$. Additionally, we cannot change any of the exponents without increasing the number again. Therefore, the minimum possible number with 6^6 factors is $2^8 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$

We now have to find greatest prime factor of the sum of the factors of this number. Now, we use the formula for the sum of the factors of a number. Going back to our example of ${p_1}^{a_1} \cdot {p_2}^{a_2} \cdots {p_n}^{a_n}$, where p_i are the prime factors and and a_i are their exponents, we can express the sum of its factors as

$$(1 + p_1 + p_1^2 + ... + p_1^{a_1 - 1} + p_1^{a_1})(1 + p_2 + p_2^2 + ... + p_2^{a_2 - 1} + p_2^{a_2}) \cdots (1 + p_n + p_n^2 + ... + p_n^{a_n - 1} + p_n^{a_n}).$$

nm In order to see how this works, we note that every single factor of ${p_1}^{a_1} \cdot {p_2}^{a_2} \cdots {p_n}^{a_n}$ can be expressed as

$$p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n},$$

where the x_i is any positive integer less than a_i. So, once distributed, our formula becomes the sum of all combinations of $p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}$ and therefore is the sum of all the number's factors. Using the formula on $2^8 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$, we get

$$(1 + 2 + 2^2 + 2^3 + \cdots + 2^8)(1 + 3 + 3^2 + \cdots + 3^5)(1 + 5 + 5^2)(1 + 7 + 7^2)(1 + 11 + 11^2)(1 + 13)(1 + 17)(1 + 19)(1 + 23)(1 + 29)$$

We have $1 + 2 + 2^2 + 2^3 + \cdots + 2^8 = 511 = 7 \cdot 73$. We check the other factors for larger prime factors.

$$1 + 3 + 3^2 + \cdots + 3^5 = 364 = 2^2 \cdot 7 \cdot 13,$$
$$1 + 5 + 5^2 = 31,$$
$$1 + 7 + 7^2 = 57 = 3 \cdot 19,$$
$$\text{and } 1 + 11 + 11^2 = 133 = 7 \cdot 19.$$

(Note that we do not have to check the other factors as they are all less than 73)

None of these primes are larger, so the largest prime factor is $\boxed{73}$.

Proposed by Justin Xu

14. **Problem:** April, Mei, June, and Jason are playing a game. They take turns playing the game in this order. April begins by rolling a 12 sided dice with the names of all 12 calendar months on it; she wins if she rolls the side with her name on it, and otherwise the game continues and she passes it to the next person. Similarly, Mei (May) and then June do the same. On Jason's turn, he wins if the dice lands on any of the months July, August, September, October, and November. The probability of Jason being the first winner is $\frac{m}{n}$, where m and n are relatively prime. Find $m + n$.

    Solution: On her turn, April has a 1/12 chance of winning because there are 12 months and one (April) is a favorable outcome. Similarly, May and June also have a 1/12 chance of winning on their turn. On Jason's turn, he has a 5/12 chance of winning because there are 12 months and five of them are winning rolls.

    For some turn to occur, all of the previous turns need to have been losing. For Jason to even be able to roll, the previous three players cannot win; thus, the probability of him being able to roll is $(11/12)^3 = 1331/1728$. The probability of him rolling for a second time means that all four players must lose in addition to the first three losses, multiplying the overall probability by $(11/12)^3 \cdot 7/12 = 9317/20736$. For Jason to roll an n-th time, the initial probability of 1331/1728 must be multiplied by $(9317/20736)^{n-1}$ because they must all lose an additional $n - 1$ times.

    The probability of Jason winning on the nth turn is

$$P(n) = \frac{1331}{1728} \cdot \frac{5}{12} \cdot \left(\frac{9317}{20736}\right)^{n-1}.$$

The total probability of him winning is the sum of all of these:

$$P(1)+P(2)+P(3)+...+P(n)=\frac{6655}{20736}\left(1+\frac{9317}{20736}+\left(\frac{9317}{20736}\right)^2+\cdots\right).$$

By the formula for the sum of an infinite geometric sequence, this reduces to:

$$\frac{6655}{20736}\left(\frac{20736}{11419}\right)=\frac{6655}{11419}\Rightarrow \boxed{18074}.$$

Proposed by Raymond Gao

15. **Problem:** Let a_1, a_2, a_3, and a_4 be roots of the polynomial Mx^4+Nx^2+M. Given that M and N are both positive integers, $\gcd(M,N)=1$, and

$$\prod_{i=1}^{4}(1-2a_i)^2\left(3-\frac{2}{a_i}\right)=2374311,$$

find $M+N$.

Note 1: $2374311=3\cdot 71^2\cdot 157$.

Note 2: This problem has been voided due to incorrect number input. A corrected version of this problem has been presented here.

Solution: We may rewrite the requested product as such:

$$\prod_{i=1}^{4}(0.5-a_i)^2\left(1.5-\frac{1}{a_i}\right)=\frac{2374311}{4096}.$$

Since Mx^4+Nx^2+M is a reciprocal polynomial, the roots are reciprocals of each other. Suppose $f(x)=Mx^4+Nx^2+M$. This means that

$$\prod_{i=1}^{4}(0.5-a_i)^2\left(1.5-\frac{1}{a_i}\right)=f(0.5)^2f(1.5).$$

Plugging these in, we get

$$\left(\frac{M+4N+16M}{16}\right)^2\left(\frac{81M+36N+16M}{16}\right)=\frac{2374311}{4096}.$$

This means that

$$(17M+4N)^2(97M+36N)=2374311=3\cdot 71^2\cdot 157.$$

Since both M and N are positive integers, we have that

$$17M+4N=71$$
$$97M+36N=3\cdot 157=471.$$

Solving for M and N, we get that $(M,N)=(3,5)$ which means that the answer is $3+5=\boxed{8}$.

Proposed by Tanish Parida