Acton-Boxborough Math Competition Online Contest

Saturday, December 14 — Sunday, December 15, 2024

Contest Rules and Format

The 2024 December Contest consists of 15 problems worth 1 point each — each with an answer between 0 and 100,000. The contest window is

Saturday, December 14 to Sunday, December 15.

The following rules supersede any others regarding the Online Contests.

- To submit your responses, fill out a form on our website, `abmathcompetition.wordpress.com`. Include your email, name, school, grade (we require this information for prize purposes).
- Submissions will close at 10:00PM Eastern Standard Time, on the final day of the round.
- Computational aids such as WolframAlpha, calculators, abaci, and other related materials are prohibited. Graphing technology such as GeoGebra, Desmos, or a graphing calculator are also prohibited. Drawing aids such as rulers and protractors are permissible.
- You may only work individually on the problems—consultation with others is not permitted.
- You may take as much time as you wish during the contest window.
- Ties will be broken by the most difficult problem solved, determined by the number of correct answers to the question. If problem A is solved by a contestants, and problem B is solved by b contestants, with $a < b$, then problem A is said to be "more difficult" than B.

Awards and Prizes

- Top scoring contestants within each round will have their names posted on our website.
- Scores will be emailed to contestants within one week of the contest's end.
- Top scoring contestants across all three rounds will receive prizes.
- Physical prizes (such as AoPS gift certificates, merchandise, calculators, etc.) will be given to top scorers if they attend the ABMC onsite contest.

Thanks to our Sponsors!

Good luck!

Problems

1. What is $(1+3+5+7+...+2021+2023+2025)-(2+4+6+8+...+2020+2022+2024)$?

2. What is the smallest positive integer that is not a factor of $30!$?

   Note: $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$.

3. Mr. Xiang is bored! He fills a marble bag with 2 white, 6 blue, and 3 black marbles. Then, he draws 2 marbles without replacement. The probability Mr. Xiang draws at least 1 white marble can be written as $\frac{a}{b}$, where a and b are relatively prime. Find $a+b$.

4. A regular polygon P has n sides and interior angles of a degrees. Another regular polygon Q with $2n$ sides has interior angles of $a+5$ degrees. Find $a+n$.

5. How many positive integers n less than 1000 satisfy the condition that n and $n+1$ share none of the same digits?

6. In a group of five people, the probability that at least four people were born on the same day of the week can be expressed as $\frac{p}{q}$, where p and q are relatively prime. Find $p+q$.

7. Lin is eating a circular pizza. He eats a slice (a sector) that takes up $x\%$ of the pizza. He starts to play with his food (which you shouldn't do) and made a cone shape with the remaining pizza by connecting the edges where the cuts were made. Lin figured out that the cone had a height of 1 foot and a base radius of 9 inches. Find x.

8. Four selfish & adventurous friends Adam, Bob, Casey, and Daniel discover a hidden treasure chest containing a total of 100 gold coins. They decide to distribute the coins among themselves. However, they realize that the sum of the squares of the number of coins each person receives must be 2500; otherwise, they will be cursed by the treasure. They collectively decide that Adam will not get more coins than anyone else. What is the smallest possible number of coins that Adam could receive under these conditions?

9. Anirudh is trying to permute the numbers 1 through 10. He has a few rules, however.

   - The sum of the first three numbers must be 16.
   - The product of the last two numbers must be 12.
   - The number 1 must be in the sixth position.

   How many ways are there for Anirudh to permute the numbers?

10. Let $d(x)$ denote the number of positive integer divisors of x. Compute the sum of all values of x such that $0 < x \leq 243$ and $d(x)+1 = d(x+1)$.

11. Let n be a positive integer. Define J and K such that

    $$J = n \cdot \frac{2025}{2022}, \qquad K = n \cdot \frac{2022}{2025}.$$

    It is known that J is divisible by 2023 and K is divisible by 2024. For the minimum value of n, the prime factorization of the greatest common divisor of J and K can be written as

    $$\gcd(J,K) = (p_1)(p_2)(p_3)(p_4)^2(p_5),$$

    where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Find $p_1+p_2+p_3+p_4+p_5$.

12. A line segment in the coordinate plane starts with endpoints at $(0,0)$ and $(0,1)$. Then, it moves according to the following algorithm:

    On the first iteration, it is rotated about the origin $90°$ clockwise such that it now has endpoints at $(0,0)$ and $(1,0)$. Then, its length is increased by 1, so that it now has endpoints at $(0,0)$ and $(0,2)$.

    On the second iteration, it is rotated $45°$ counterclockwise, and then its length is again increased by 1.

    In general, on the nth iteration, the segment is rotated about the origin $\frac{90}{2^n}^{\circ}$ in alternating directions, then extended by 1 unit away from the origin.

    As this process repeats forever, the total area swept out by the line segment can be expressing as $\frac{a}{b}\pi$, where a and b are relatively prime. Find $a+b$.

13. Infinite particles are shot in every direction from the center of an equilateral triangle. When a particle hits a side, it reflects according to the law of reflection. The particle travels at 2 centimeters per second and stops when it reaches a corner.

    How many particles will reach a corner within eight seconds?

    Clarification: Although there are infinite particles, only one particle is being shot in any given direction from the center.

    Note: The *law of reflection* states that the angle of incidence (the angle the particle's start path makes with the triangle's side) is congruent to the angle of reflection (the angle the particle's reflected path makes with the triangle's side.)

14. Consider the following infinite product P:

$$P = \prod_{n \in S} \sqrt[n]{n} = \sqrt[1]{1} \cdot \sqrt[5]{5} \cdot \sqrt[7]{7} \cdot \sqrt[11]{11} \cdot \sqrt[25]{25} \cdot \sqrt[35]{35} \cdots$$

    where S is the infinite set of positive integers with exclusively 5, 7, and/or 11 in its prime factorization. In other words, the elements in S are expressed as $5^x \cdot 7^y \cdot 11^z$ for all nonnegative integers x, y, z.

    P can be expressed as $5^a \cdot 7^b \cdot 11^c$ for rational numbers a, b, and c. If $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, and $c = \frac{c_1}{c_2}$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are positive integers where $\gcd(a_1, a_2) = 1$, $\gcd(b_1, b_2) = 1$, $\gcd(c_1, c_2) = 1$. Find $a_1 + a_2 + b_1 + b_2 + c_1 + c_2$.

15. Alice writes a horizontal line of 10 2's on a blackboard. Bob then places one of the four operators $(+, -, \times, \div)$ in the 9 gaps in between, labeled as "?". The expected value of the final computation can be expressed as

$$E = a \cdot \left(\frac{b}{c}\right)^{10} - d,$$

    where a, b, c, and d are positive integers such that b and c are relatively prime. Find $a+b+c+d$.

    A visualization of the blackboard is shown below.

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    Note: Alice considers the order of operations as she calculates the final computation.