Acton-Boxborough Math Competition Online Contest Solutions

Saturday, December 14 — Sunday, December 15, 2024

1. **Problem:** What is $(1+3+5+7+...+2021+2023+2025)-(2+4+6+8+...+2020+2022+2024)$?

   Solution: Notice that the expression
   $$(1+3+5+7+...+2021+2023+2025)-(2+4+6+8+...+2020+2022+2024)$$
   can be rewritten as:
   $$1+(3-2)+(5-4)+...+(2025-2024).$$
   There are 1013 terms that equal 1 in this list, so the answer is $1 \cdot 1013 = \boxed{1013}$.

   Proposed by Shubham Kulkarni

2. **Problem:** What is the smallest positive integer that is not a factor of 30!?
   Note: $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$.

   Solution: Because 30! is equal to the product of all the positive integers less than or equal to 30, we know that the prime factorization includes all the primes less than 30. Therefore, the smallest positive integer that is not a factor of 30! is $\boxed{31}$, the smallest prime greater than 30.

   Proposed by Aarush Kulkarni

3. **Problem:** Mr. Xiang is bored! He fills a marble bag with 2 white, 6 blue, and 3 black marbles. Then, he draws 2 marbles without replacement. The probability Mr. Xiang draws at least 1 white marble can be written as $\frac{a}{b}$, where a and b are relatively prime. Find $a+b$.

   Solution: We are asked to find the probability that **at least** 1 white marble is drawn, we can calculate the 1 minus the probability that **no** white marbles are drawn.
   Because there are 9 marbles that are not white and there are 11 marbles in total, the probability of drawing no white marbles is
   $$\frac{\binom{9}{2}}{\binom{11}{2}} = \frac{36}{55}.$$
   Therefore, the probability of drawing at least 1 white marble is $1 - 36/55 = 19/55$. Our final answer is $19+55 = \boxed{74}$.

   Proposed by Christopher Zhang

4. **Problem:** A regular polygon P has n sides and interior angles of a degrees. Another regular polygon Q with $2n$ sides has interior angles of $a+5$ degrees. Find $a+n$.

   Solution: We know that the interior angles of a polygon with n sides is given from
   $$a_n = \frac{180(n-2)}{n}.$$
   We know that $a_{2n} = a_n + 5$, so we solve the equation
   $$\frac{180(2n-2)}{2n} = \frac{180(n-2)}{n} + 5.$$
   Multiplying both sides by $2n$ and simplifying, we have:
   $$\begin{aligned} 180(2n-2) &= 360(n-2) + 10n \\ 360n - 360 &= 360n - 720 + 10n \\ 10n &= 360. \end{aligned}$$
   Therefore, polygon P has $n = 36$ sides and interior angles of $a = 170°$. Our final answer is $36 + 170 = \boxed{206}$.

   Proposed by Nathan Tan

5. **Problem:** How many positive integers n less than 1000 satisfy the condition that n and $n+1$ share none of the same digits?

   Solution: There are three cases to consider.

   - **Case 1: One-digit numbers.** All single-digit numbers n have different digits from $n+1$. Nine positive integers satisfy this case.
   - **Case 2: Two-digit numbers.** All numbers that don't end in 9 can't satisfy the condition, since the ten's digit will not change if you add 1. For those that end in 9, the numbers 19, 29, ..., 79 and 99 satisfy the condition. We must exclude 89 since it shares a 9 with 90. Eight integers satisfy this case.
   - **Case 3: Three-digit numbers.** All numbers that don't end in 99 can't satisfy the condition, since the hundreds digit will not change. For those that end in 99, the numbers 199, 299, ..., 799, and 999 satisfy the condition. We must exclude 899 since it shares a 9 with 900. Eight integers satisfy this case.

   Combining these cases, there are $9+8+8 = \boxed{25}$ integers that satisfy the condition.

   Proposed by Ayaan Garg

6. **Problem:** In a group of five people, the probability that at least four people were born on the same day of the week can be expressed as $\frac{p}{q}$, where p and q are relatively prime. Find $p+q$.

   Solution: There are 7 possible days of the week each person can be born on. In total, there are 7^5 possible combinations of birth days for the group.

   Now, we can split the problem into two cases:

   - **Case 1: Exactly 4 people are born on the same day.** There are 7 choices for the day where these 4 people are born, and there are $\binom{5}{4}$ ways to choose 4 out of the 5 people to be born on the same day, and 6 ways to choose the day where the 5th person is born. In total, there are $7 \cdot \binom{5}{4} \cdot 6 = 210$ ways for this case.
   - **Case 2: All 5 people are born on the same day.** There are 7 choices for the day where the 5 people are born, and no other variations are possible.

   Combining these cases, there are 217 ways for at least 4 people to be born on the same day. Finally, dividing this by the total possible outcomes yields

   $$\frac{p}{q} = \frac{217}{7^5} = \frac{31}{2401}.$$

   Our final answer is $31 + 2401 = \boxed{2432}$.

   Proposed by Daniel Ren

7. **Problem:** Lin is eating a circular pizza. He eats a slice (a sector) that takes up x% of the pizza. He starts to play with his food (which you shouldn't do) and made a cone shape with the remaining pizza by connecting the edges where the cuts were made. Lin figured out that the cone had a height of 1 foot and a base radius of 9 inches. Find x.

   Solution: Let's first consider how the remaining pizza becomes a cone. When you fold a sector into a cone, the remaining circumference of the pizza becomes the circular base of the cone. Additionally, the radius of the pizza becomes the slant height of the cone; that is, the distance from the top point on the cone to a point on the edge of the base.

Since the base has a radius of 9 inches, we know that the circumference of the base is 18π. Since the base is the same as the circumference of the remaining pizza, we can write

$$\left(1 - \frac{x}{100}\right) \cdot 2\pi r = 18\pi \qquad \Rightarrow \qquad (100 - x)r = 900,$$

where r is the radius of the pizza.

Now let's look at the slant height of the cone. We know that the altitude of the cone is 1 foot (or 12 inches), and the radius of the base is 9 inches. The slant height is the hypotenuse of a right triangle with legs as the radius and the altitude. So, by the Pythagorean Theorem, we know the slant height is 15. As discussed before, this slant height is equal to the radius, so we know $r = 15$.

Plugging this into our first equation, we get

$$(100 - x) \cdot 15 = 900 \Rightarrow x = \boxed{40}.$$

Proposed by Iris Shi

8. **Problem:** Four selfish & adventurous friends Adam, Bob, Casey, and Daniel discover a hidden treasure chest containing a total of 100 gold coins. They decide to distribute the coins among themselves. However, they realize that the sum of the squares of the number of coins each person receives must be 2500; otherwise, they will be cursed by the treasure. They collectively decide that Adam will not get more coins than anyone else. What is the smallest possible number of coins that Adam could receive under these conditions?

   Solution: Let the number of coins received by Adam, Bob, Casey, and Daniel be a, b, c, and d respectively. We are given the following conditions and asked to minimize a.

$$\begin{cases} a + b + c + d = 100 \\ a^2 + b^2 + c^2 + d^2 = 2500 \\ a \leq b, c, d \end{cases}$$

   Let $b = a + x$, $c = a + y$, and $d = a + z$, where $x, y, z \geq 0$. Substituting b, c, and d into Equation (1), we have:

$$\begin{aligned} a + (a + x) + (a + y) + (a + z) &= 100 \\ x + y + z &= 100 - 4a \qquad (4) \end{aligned}$$

   Substituting b, c, and d into Equation (2), we have

$$\begin{aligned} a^2 + (a + x)^2 + (a + y)^2 + (a + z)^2 &= 2500 \\ a^2 + (a^2 + 2ax + x^2) + (a^2 + 2ay + y^2) + (a^2 + 2az + z^2) &= 2500 \\ 4a^2 + 2a(x + y + z) + (x^2 + y^2 + z^2) &= 2500 \qquad (5) \end{aligned}$$

   From Equation (4), we know that $x + y + z = 100 - 4a$. Substituting this into Equation (5), we have

$$\begin{aligned} 4a^2 + 2a(100 - 4a) + (x^2 + y^2 + z^2) &= 2500 \\ 4a^2 + 200a - 8a^2 + x^2 + y^2 + z^2 &= 2500 \\ -4a^2 + 200a + (x^2 + y^2 + z^2) &= 2500 \qquad (6) \end{aligned}$$

   Rearranging to express the sum of squares of x, y, z, we have

$$x^2 + y^2 + z^2 = 4a^2 - 200a + 2500 \qquad (7)$$

Given that x, y, z are real, the sum $x^2 + y^2 + z^2$ must be non-negative. Therefore, the right-hand side of Equation (7) must be positive. Factoring, we have

$$\begin{aligned} 4a^2 - 200a + 2500 &\geq 0 \\ a^2 - 50a + 625 &\geq 0 \\ (a-25)^2 &\geq 0. \end{aligned}$$

Thus, a is minimized at $\boxed{25}$.

Solution 2 (QM-AM): The QM-AM inequality with $n = 4$ states that for 4 positive numbers x_1, x_2, x_3, and x_4, we have

$$\sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{4}} \geq \frac{x_1 + x_2 + x_3 + x_4}{4},$$

with equality occurring when $x_1 = x_2 = x_3 = x_4$. We compute the arithmetic mean and quadratic mean using the conditions given. Like before, we are given the following conditions and asked to minimize a.

$$\begin{cases} a + b + c + d = 100 \\ a^2 + b^2 + c^2 + d^2 = 2500 \\ a \leq b,\ a \leq c,\ a \leq d \end{cases}$$

By QM-AM, we have

$$\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} \geq \frac{a + b + c + d}{4}.$$

From our first constraint, we have

$$\begin{aligned} \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} &\geq \frac{100}{4} \\ a^2 + b^2 + c^2 + d^2 &\geq 2500. \end{aligned}$$

In other words, 2500 is the minimum value of $a^2 + b^2 + c^2 + d^2$. The only conditions that minimize $a^2 + b^2 + c^2 + d^2$ is when $a = b = c = d$: when everyone gets the same amount of gold. Therefore, Adam will get $100/4 = \boxed{25}$ coins.

(Note: Seems like the treasure chest was fair after all!)

Proposed by Omar Graia

9. **Problem:** Anirudh is trying to permute the numbers 1 through 10. He has a few rules, however.

   - The sum of the first three numbers must be 16.
   - The product of the last two numbers must be 12.
   - The number 1 must be in the sixth position.

   How many ways are there for Anirudh to permute the numbers?

   Solution: There are four ways to permute that last two numbers, namely (2,6); (3,4); (4,3); and (6,2).

   Now, we can split the problem into two cases:

- **Case 1: The last two numbers are 2 and 6** (In some order)
  If this is the case, the numbers 1, 2, and 6 are not present in the first three numbers. The triplets (3,4,9); (3,5,8); and (4,5,7) and their permutations are the only triplets that satisfy the first condition. Since 1 is fixed at the 6th position, there are 4! ways to arrange the remaining 4 numbers. There are a total of $2 \cdot 3 \cdot 6 \cdot 4! = 864$ ways for this case.
- **Case 2: The last two numbers are 3 and 4** (In some order)
  If this is the case, the numbers 1, 3, and 4 are not present in the first three numbers. The triplets (2,5,8) and (2,6,7) and their permutations are the only triplets that satisfy the first condition. Since 1 is fixed at the 6th positions, there are 4! ways to arrange the remaining 4 numbers. There are a total of $2 \cdot 2 \cdot 6 \cdot 4! = 576$ ways for this case.

In total, there are $864 + 576 = \boxed{1440}$ ways for Anirudh to permute the numbers.

Proposed by Anirudh Pulugurtha

10. **Problem:** Let $d(x)$ denote the number of positive integer divisors of x. Compute the sum of all values of x such that $0 < x \leq 243$ and $d(x) + 1 = d(x + 1)$.

    Solution: Since we have $d(x) + 1 = d(x + 1)$, we know that either $d(x)$ or $d(x + 1)$ must be odd. This means that either x or $x + 1$ must have an odd number of divisors. In order for a positive integer to have an odd number of divisors, it must be a perfect square, since otherwise we would be able to pair up factors and get an even number. This means that in order for x to satisfy the condition, it must be either a perfect square or 1 less than a perfect square. So, we can simply check all such numbers between 1 and 243 and see if they satisfy the condition.

    Checking 1, it has only 1 factor, while 2 has 2, so 1 is a value. Moving on to 3, it has 2 factors while 4 has 3, so it also works. 4 has 3 factors, but 5 has 2, so 4 does not work. Continuing to check all of these numbers, we find that all the values of x that satisfy the condition are $1, 3, 9, 15, 25, 63, 121$, and 195. Adding all these numbers together, we get $\boxed{432}$ as the final answer.

    Proposed by Bryan Li

11. **Problem:** Let n be a positive integer. Define J and K such that

    $$J = n \cdot \frac{2025}{2022}, \qquad K = n \cdot \frac{2022}{2025}.$$

    It is known that J is divisible by 2023 and K is divisible by 2024. For the minimum value of n, the prime factorization of the greatest common divisor of J and K can be written as

    $$\gcd(J, K) = (p_1)(p_2)(p_3)(p_4)^2(p_5),$$

    where p_1, p_2, p_3, p_4, and p_5 are prime numbers. Find $p_1 + p_2 + p_3 + p_4 + p_5$.

    Solution: First, factor the given constants:

    $$2022 = 2 \cdot 3 \cdot 337, \quad 2023 = 7 \cdot 17^2, \quad 2024 = 2^3 \cdot 11 \cdot 23, \quad 2025 = 3^4 \cdot 5^2.$$

    Rewrite J and K in terms of these prime factorizations:

    $$J = n \cdot \frac{2025}{2022} = n \cdot \frac{3^4 \cdot 5^2}{2 \cdot 3 \cdot 337} = \frac{n \cdot 3^3 \cdot 5^2}{2 \cdot 337}.$$

    $$K = n \cdot \frac{2022}{2025} = n \cdot \frac{2 \cdot 3 \cdot 337}{3^4 \cdot 5^2} = \frac{n \cdot 2 \cdot 337}{3^3 \cdot 5^2}.$$

    Thus:

    $$J = \frac{n \cdot 3^3 \cdot 5^2}{2 \cdot 337}, \quad K = \frac{n \cdot 2 \cdot 337}{3^3 \cdot 5^2}.$$

    The following constraints hold:

(a) J must be divisible by $2023 = 7 \cdot 17^2$. For J to be an integer, n must include factors to cancel out the denominator $2 \cdot 337$. Let n be divisible by $2 \cdot 337 = 674$. Set $n = 674m$:

$$J = \frac{674m \cdot 3^3 \cdot 5^2}{674} = 675m.$$

Now $J = 675m$ must be divisible by $2023 = 7 \cdot 17^2$. Since $675 = 3^3 \cdot 5^2$ shares no common factors with 2023, m must be divisible by 2023. Let $m = 2023m_1$. Thus from J's conditions:

$$n = 674 \cdot 2023 \cdot m_1.$$

(b) K must be divisible by $2024 = 2^3 \cdot 11 \cdot 23$. For K to be an integer:

$$K = \frac{n \cdot 674}{3^3 \cdot 5^2} = \frac{n \cdot 674}{675}.$$

Since $675 = 3^3 \cdot 5^2$ and 674 and 675 share no factors, n must be divisible by 675 to cancel the denominator. Set $n = 675q$, then:

$$K = \frac{675q \cdot 674}{675} = 674q.$$

For K to be divisible by 2024, $674q$ must be divisible by 2024. Factor $674 = 2 \cdot 337$. We have:

$$2 \cdot 337 \cdot q \text{ divisible by } 2^3 \cdot 11 \cdot 23.$$

Cancel one factor of 2:

$$337q \text{ must be divisible by } 2^2 \cdot 11 \cdot 23 = 4 \cdot 11 \cdot 23 = 1012.$$

Since 337 is prime and does not share factors with $1012 = 2^2 \cdot 11 \cdot 23$, q must be divisible by 1012. Thus from K's conditions:

$$n = 675 \cdot 1012 \cdot r.$$

Combining both sets of conditions, n must be a common multiple of $674 \cdot 2023$ and $675 \cdot 1012$. The minimal n is the least common multiple of these two products. Factor each component fully:

$$674 \cdot 2023 = (2 \cdot 337) \cdot (7 \cdot 17^2) = 2 \cdot 7 \cdot 17^2 \cdot 337$$
$$675 \cdot 1012 = (3^3 \cdot 5^2) \cdot (2^2 \cdot 11 \cdot 23) = 2^2 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 23$$

The least common multiple must include all prime factors at their highest powers:

$$\text{lcm}(674 \cdot 2023,\ 675 \cdot 1012) = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 337.$$

This minimal n satisfies both conditions. With the minimal n, we have:

$$J = \frac{n \cdot 3^3 \cdot 5^2}{2 \cdot 337}, \quad K = \frac{n \cdot 2 \cdot 337}{3^3 \cdot 5^2}.$$

Substitute $n = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 337$: For J:

$$J = \frac{(2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 337)(3^3 \cdot 5^2)}{2 \cdot 337}.$$

Canceling 2 and 337:

$$J = 2 \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23.$$

For K:

$$K = \frac{(2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 337)(2 \cdot 337)}{3^3 \cdot 5^2}.$$

Canceling 3^3 and 5^2:

$$K = 2^3 \cdot 7 \cdot 11 \cdot 17^2 \cdot 23 \cdot 337^2.$$

The prime factorization of J is $2^1 \cdot 3^6 \cdot 5^4 \cdot 7^1 \cdot 11^1 \cdot 17^2 \cdot 23^1$, and the prime factorization of K is $2^3 \cdot 7^1 \cdot 11^1 \cdot 17^2 \cdot 23^1 \cdot 337^2$. The common prime factors (taking the minimum exponents) are:

$$\gcd(J, K) = 2^1 \cdot 7^1 \cdot 11^1 \cdot 17^2 \cdot 23^1.$$

This matches the form:

$$\gcd(J, K) = (p_1)(p_2)(p_3)(p_4)^2(p_5)$$

with primes $p_1 = 2$, $p_2 = 7$, $p_3 = 11$, $p_4 = 17$, and $p_5 = 23$. We need $p_1 + p_2 + p_3 + p_4 + p_5$, so our final answer is $2 + 7 + 11 + 17 + 23 = \boxed{60}$.

Proposed by Daniel Ren

12. **Problem:** A line segment in the coordinate plane starts with endpoints at $(0, 0)$ and $(0, 1)$. Then, it moves according to the following algorithm:

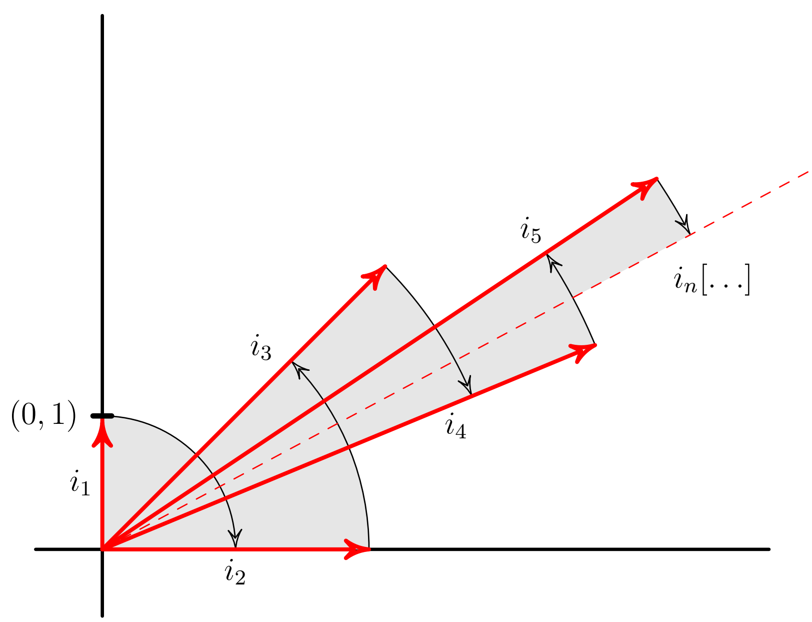
    On the first iteration, it is rotated about the origin $90°$ clockwise such that it now has endpoints at $(0, 0)$ and $(1, 0)$. Then, its length is increased by 1, so that it now has endpoints at $(0, 0)$ and $(2, 0)$.

    On the second iteration, it is rotated $45°$ counterclockwise, and then its length is again increased by 1.

    In general, on the nth iteration, the segment is rotated about the origin $\frac{90}{2^n}^{\circ}$ in alternating directions, then extended by 1 unit away from the origin.

    As this process repeats forever, the total area swept out by the line segment can be expressing as $\frac{a}{b}\pi$, where a and b are relatively prime. Find $a + b$.

    Solution: The following diagram represents the problem:



When the segment swings from i_1 to i_2, the segment sweeps an area of $\left(\frac{90}{360} \cdot 1^2\right)\pi = \frac{1}{4}\pi$.

Next, when the segment increases in length and swings from i_2 to i_3, the segment sweeps an additional area of $\frac{45}{360} \cdot (2^2 - 1^2)\pi = \frac{3}{8}\pi$ on top of the original area of $\frac{1}{4}\pi$ that was swept out previously.

Again, when the segment increases in length and swings from i_3 to i_4, the segment sweeps an additional area of $\frac{22.5}{360} \cdot (3^2 - 2^2)\pi = \frac{5}{16}\pi$.

When the segment increases in length and swings from i_4 to i_5, we notice that the segment sweeps an additional area of $\frac{11.25}{360} \cdot (4^2 - 3^2)\pi = \frac{7}{32}\pi$.

We quickly notice a pattern here. In general, when the segment sweeps from i_{n-1} to i_n, the segment sweeps out an additional area of $\left(\frac{2n-3}{2^n}\right)\pi$.

Thus, the total area that the segment sweeps out is

$$\frac{1}{4}\pi + \frac{3}{8}\pi + \frac{5}{16}\pi + \frac{7}{32}\pi + \cdots + \left(\frac{2n-3}{2^n}\right)\pi + \cdots.$$

Notice that this is an infinite arithmetico-geometric series. Let

$$S = \frac{1}{4}\pi + \frac{3}{8}\pi + \frac{5}{16}\pi + \frac{7}{32}\pi + \cdots + \left(\frac{2n-3}{2^n}\right)\pi + \cdots.$$

Then,

$$\frac{1}{2}S = \frac{1}{8}\pi + \frac{3}{16}\pi + \frac{5}{32}\pi + \frac{7}{64}\pi + \cdots + \left(\frac{2n-3}{2^{n+1}}\right)\pi + \cdots.$$

Therefore,

$$\begin{aligned} S - \frac{1}{2}S &= \left(\frac{1}{4}\pi + \frac{3}{8}\pi + \frac{5}{16}\pi + \frac{7}{32}\pi + \cdots + \left(\frac{2n-3}{2^n}\right)\pi + \cdots\right) \\ &\quad - \left(\frac{1}{8}\pi + \frac{3}{16}\pi + \frac{5}{32}\pi + \frac{7}{64}\pi + \cdots + \left(\frac{2n-3}{2^{n+1}}\right)\pi + \cdots\right) \\ &= \frac{1}{4}\pi + \left(\frac{2}{8}\pi + \frac{2}{16}\pi + \frac{2}{32}\pi + ...\right) \\ &= \frac{1}{4}\pi + \frac{\frac{1}{4}\pi}{1 - \frac{1}{2}} \\ &= \frac{3}{4}\pi \end{aligned}$$

Since $S - \frac{1}{2}S = \frac{1}{2}S = \frac{3}{4}\pi$, we see that $S = \frac{3}{2}\pi$, which is the total area that the segment sweeps out. Our final answer is $3 + 2 = \boxed{5}$.

Proposed by Bryan Li

13. **Problem:** Infinite particles are shot in every direction from the center of an equilateral triangle with a side length of $2\sqrt{3}$ cm. When a particle hits a side, it reflects according to the law of reflection. The particle travels at 2 centimeters per second and stops when it reaches a corner.

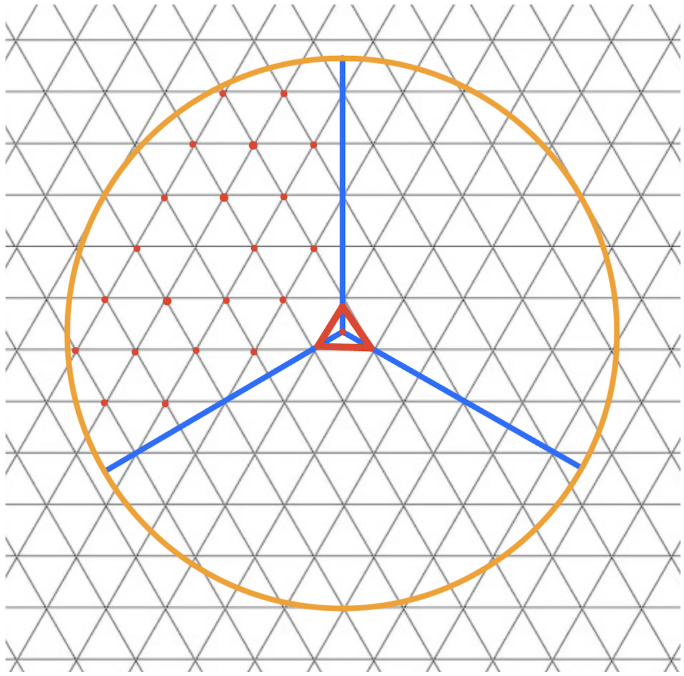
    How many particles will reach a corner within eight seconds?

    Clarification: Although there are infinite particles, only one particle is being shot in any given direction from the center.

    Note: The *law of reflection* states that the angle of incidence (the angle the particle's start path makes with the triangle's side) is congruent to the angle of reflection (the angle the particle's reflected path makes with the triangle's side.)

    Solution: Every time a particle hits an edge, it bounces back according to the law of reflection. Instead of imagining it as bouncing back, we can reflect the whole triangle across the side that the particle bounces off of. This makes the path of the particle a straight line.

    Extending this idea, we can see that instead of having to deal with any reflections, we can just turn our grid into a triangular one and every particle moves in a straight line. The farthest a particle could go is 16 centimeters. We can draw a circle around the center to see how far the particles travel. Now we can count how many particles end up at a corner within that circle. This gets us $\boxed{66}$ directions where the particle reaches a corner. The diagram below shows a third of the diagram where the red dots are the places where a particle can reach a corner.



Proposed by Tanish Parida

14. **Problem:** Consider the following infinite product P:

$$P = \prod_{n \in S} \sqrt[n]{n} = \sqrt[1]{1} \cdot \sqrt[5]{5} \cdot \sqrt[7]{7} \cdot \sqrt[11]{11} \cdot \sqrt[25]{25} \cdot \sqrt[35]{35} \cdots$$

where S is the infinite set of positive integers with exclusively $5, 7,$ and/or 11 in its prime factorization. In other words, the elements in S are expressed as $5^x \cdot 7^y \cdot 11^z$ for all nonnegative integers x, y, z.

P can be expressed as $5^a \cdot 7^b \cdot 11^c$ for rational numbers a, b, and c. If $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$, and $c = \frac{c_1}{c_2}$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are positive integers where $\gcd(a_1, a_2) = 1$, $\gcd(b_1, b_2) = 1$, $\gcd(c_1, c_2) = 1$. Find $a_1 + a_2 + b_1 + b_2 + c_1 + c_2$.

Solution: Since each term in the product P is of the form:

$$\sqrt[n]{n} = n^{1/n} = (5^x \cdot 7^y \cdot 11^z)^{1/(5^x \cdot 7^y \cdot 11^z)},$$

the infinite product becomes:

$$P = \prod_{x=0} \prod_{y=0} \prod_{z=0} (5^x \cdot 7^y \cdot 11^z)^{1/(5^x \cdot 7^y \cdot 11^z)}.$$

This can be separated into the product of powers of each prime:

$$P = 5^{\sum_{x,y,z \geq 0} \frac{x}{5^x \cdot 7^y \cdot 11^z}} \cdot 7^{\sum_{x,y,z \geq 0} \frac{y}{5^x \cdot 7^y \cdot 11^z}} \cdot 11^{\sum_{x,y,z \geq 0} \frac{z}{5^x \cdot 7^y \cdot 11^z}}.$$

Therefore, we identify:

$$a = \sum_{x,y,z \geq 0} \frac{x}{5^x \cdot 7^y \cdot 11^z}, \quad b = \sum_{x,y,z \geq 0} \frac{y}{5^x \cdot 7^y \cdot 11^z}, \quad c = \sum_{x,y,z \geq 0} \frac{z}{5^x \cdot 7^y \cdot 11^z}.$$

Notice that the sums can be factored as products of geometric series:

$$a = \left(\sum_{x=0} \frac{x}{5^x}\right) \cdot \left(\sum_{y=0} \frac{1}{7^y}\right) \cdot \left(\sum_{z=0} \frac{1}{11^z}\right),$$

$$b = \left(\sum_{x=0} \frac{1}{5^x}\right) \cdot \left(\sum_{y=0} \frac{y}{7^y}\right) \cdot \left(\sum_{z=0} \frac{1}{11^z}\right),$$

$$c = \left(\sum_{x=0} \frac{1}{5^x}\right) \cdot \left(\sum_{y=0} \frac{1}{7^y}\right) \cdot \left(\sum_{z=0} \frac{z}{11^z}\right).$$

We evaluate each geometric series and related sums using known formulas:

$$\sum_{n=0} r^n = \frac{1}{1-r}, \quad \sum_{n=0} nr^n = \frac{r}{(1-r)^2}.$$

Calculating each sum individually:

$$\sum_{x=0} \frac{x}{5^x} = \frac{\frac{1}{5}}{\left(1-\frac{1}{5}\right)^2} = \frac{\frac{1}{5}}{\left(\frac{4}{5}\right)^2} = \frac{1}{5} \cdot \frac{25}{16} = \frac{5}{16},$$

$$\sum_{y=0} \frac{1}{7^y} = \frac{1}{1-\frac{1}{7}} = \frac{7}{6},$$

$$\sum_{z=0} \frac{1}{11^z} = \frac{1}{1-\frac{1}{11}} = \frac{11}{10},$$

$$\sum_{y=0} \frac{y}{7^y} = \frac{\frac{1}{7}}{\left(1-\frac{1}{7}\right)^2} = \frac{\frac{1}{7}}{\left(\frac{6}{7}\right)^2} = \frac{1}{7} \cdot \frac{49}{36} = \frac{7}{36},$$

$$\sum_{z=0} \frac{z}{11^z} = \frac{\frac{1}{11}}{\left(1-\frac{1}{11}\right)^2} = \frac{\frac{1}{11}}{\left(\frac{10}{11}\right)^2} = \frac{1}{11} \cdot \frac{121}{100} = \frac{11}{100},$$

$$\sum_{x=0} \frac{1}{5^x} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}.$$

Substituting the evaluated sums back into the expressions for a, b, and c:

$$a = \left(\frac{5}{16}\right) \cdot \left(\frac{7}{6}\right) \cdot \left(\frac{11}{10}\right) = \frac{385}{960} = \frac{77}{192},$$

$$b = \left(\frac{5}{4}\right) \cdot \left(\frac{7}{36}\right) \cdot \left(\frac{11}{10}\right) = \frac{385}{1440} = \frac{77}{288},$$

$$c = \left(\frac{5}{4}\right) \cdot \left(\frac{7}{6}\right) \cdot \left(\frac{11}{100}\right) = \frac{385}{2400} = \frac{77}{480}.$$

Thus, the infinite product P can be expressed as:

$$P = 5^{\frac{77}{192}} \cdot 7^{\frac{77}{288}} \cdot 11^{\frac{77}{480}}.$$

Our final answer is $77 + 192 + 77 + 288 + 77 + 480 = \boxed{1191}$.

Proposed by Andrew Wu

15. **Problem:** Alice writes a horizontal line of 10 2's on a blackboard. Bob then places one of the four operators $(+, -, \times, \div)$ in the 9 gaps in between, labeled as "?". The expected value of the final computation can be expressed as

$$E = a \cdot \left(\frac{b}{c}\right)^{10} - d,$$

where a, b, c, and d are positive integers such that b and c are relatively prime. Find $a + b + c + d$.

A visualization of the blackboard is shown below.

$$2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2 \ \boxed{?} \ 2.$$

Note: Alice considers the order of operations as she calculates the final computation.

Solution: This problem has been voided due to the incorrect answer input format. In this solution, instead of finding $a + b + c + d$, we will find E, the expected value of the blackboard computation.

We can do casework on the first occurrence of a $+/-$ and calculate the expected value for each case. After the first $+/-$, the expected value of the rest of the expression is 0. This is because we can always substitute a $+$ instead of a $-$ (or vice versa), which cancels out the part after the first $+/-$.

- If the first $+/-$ appears in the first $\boxed{?}$, the expected value for the entire expression is 2. This occurs with $\frac{1}{2}$ probability.
- If the first $+/-$ appears in the second $\boxed{?}$, the expected value for the entire expression is the average of 2×2 and $2 \div 2$, or $\frac{5}{2}$. This occurs with $\frac{1}{4}$ probability.
- If the first $+/-$ appears in the third $\boxed{?}$, the expected value for the entire expression is the average of $\frac{5}{2} \times 2$ and $\frac{5}{2} \div 2$, or $\frac{5}{8}$. This occurs with $\frac{1}{8}$ probability.
- We notice a pattern. In general, if the first $+/-$ appears in the kth $\boxed{?}$, the expected value will be $2\left(\frac{5}{4}\right)^{k-1}$, with a $\left(\frac{1}{2}\right)^k$ probability of occurring.
- If there are no $+/-$ at all, which occurs with $\left(\frac{1}{2}\right)^9$ probability, the expected value of the expression will be $2\left(\frac{5}{4}\right)^9$.

Our goal is to find the final expected value E:

$$E = \left[\sum_{k=1}^{9} 2\left(\frac{5}{4}\right)^{k-1}\left(\frac{1}{2}\right)^{k}\right] + 2\left(\frac{5}{4}\right)^{9}\left(\frac{1}{2}\right)^{9}.$$

With some manipulation, we have:

$$\begin{aligned} E &= \left[\sum_{k=1}^{9} 2\left(\frac{5}{4}\right)^{k-1}\left(\frac{1}{2}\right)^{k-1}\left(\frac{1}{2}\right)\right] + 2\left(\frac{5}{4}\right)^{9}\left(\frac{1}{2}\right)^{9} \\ &= \left[\sum_{k=1}^{9} \left(\frac{5}{4} \cdot \frac{1}{2}\right)^{k-1}\right] + 2\left(\frac{5}{4} \cdot \frac{1}{2}\right)^{9} \\ &= \left[\sum_{k=1}^{9} \left(\frac{5}{8}\right)^{k-1}\right] + 2\left(\frac{5}{8}\right)^{9}. \end{aligned}$$

The summation is a geometric series with $n = 9$ terms, first term $a = 1$, and ratio $r = \frac{5}{8}$. Using the formula for a geometric series:

$$\begin{aligned} S &= \frac{a(1 - r^n)}{1 - r} = \frac{1(1 - (5/8)^9}{3/8} \\ &= \frac{8}{3}\left(1 - \left(\frac{5}{8}\right)^9\right) = \frac{8}{3} - \frac{8}{3}\left(\frac{5}{8}\right)^9. \end{aligned}$$

Subtracting the final term, the expected value is

$$E = \frac{8}{3} - \frac{8}{3}\left(\frac{5}{8}\right)^9 + 2\left(\frac{5}{8}\right)^9$$
$$= \boxed{\frac{8}{3} - \frac{2}{3}\left(\frac{5}{8}\right)^9}.$$

This answer did not fit our current answer format, so the problem was voided.

Proposed by Bryan Li