# Acton-Boxborough Math Competition Online Contest

Saturday, November 16th — Sunday, November 17th, 2024

# Contest Rules and Format

The 2024 November Contest consists of 15 problems worth 1 point each — each with an answer between 0 and 100,000. The contest window is

Saturday, November 16th to Sunday, November 17th.

The following rules supersede any others regarding the Online Contests.

- To submit your responses, fill out a form on our website, `abmathcompetition.wordpress.com`. Include your email, name, school, grade (we require this information for prize purposes).
- Submissions will close at 10:00PM Eastern Standard Time, on the final day of the round.
- Computational aids such as WolframAlpha, calculators, abaci, and other related materials are prohibited. Graphing technology such as GeoGebra, Desmos, or a graphing calculator are also prohibited. Drawing aids such as rulers and protractors are permissible.
- You may only work individually on the problems—consultation with others is not permitted.
- You may take as much time as you wish during the contest window.
- Ties will be broken by the most difficult problem solved, determined by the number of correct answers to the question. If problem A is solved by $a$ contestants, and problem B is solved by $b$ contestants, with $a < b$, then problem A is said to be "more difficult" than B.

# Awards and Prizes

- Top scoring contestants within each round will have their names posted on our website.
- Scores will be emailed to contestants within one week of the contest's end.
- Top scoring contestants across all three rounds will receive prizes.
- Physical prizes (such as AoPS gift certificates, merchandise, calculators, etc.) will be given to top scorers if they attend the ABMC onsite contest.

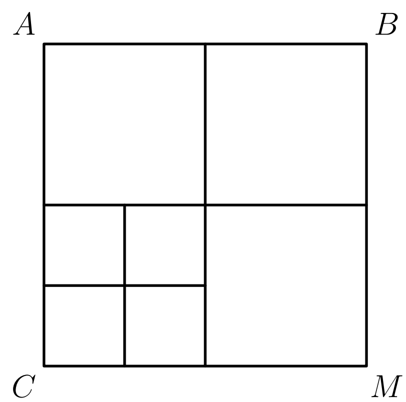
# Thanks to our Sponsors!

Good luck!

# Problems

1. Evaluate $\dfrac{2 \cdot 0 \cdot 2 \cdot 4}{2+0+2+4}$.

2. Square $ABMC$ is split into 4 squares of equal area. The square in the bottom-left corner is also split into 4 squares of equal area. Given that the area of the smallest squares shown in the diagram is 2, find the area of the largest square shown in the diagram.




3. Three teachers are grading a pile of math tests on Thanksgiving Day. Patrick can grade one every 3 minutes, Petra can grade one every 5 minutes, and Phil can grade one every 6 minutes.

   One day, each teacher begins grading at 9:00 AM and stops grading at 4:00 PM. How many tests are graded in total?

4. What is the largest prime factor of $18! + 19!$?

   **Note:** $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$.

5. I have six white socks, two black socks, and eight purple socks in my drawer. On one morning, I randomly pick two socks to wear. The probability I pick two different colored socks is $\frac{p}{q}$, where $p$ and $q$ are relatively prime. Find $p+q$.

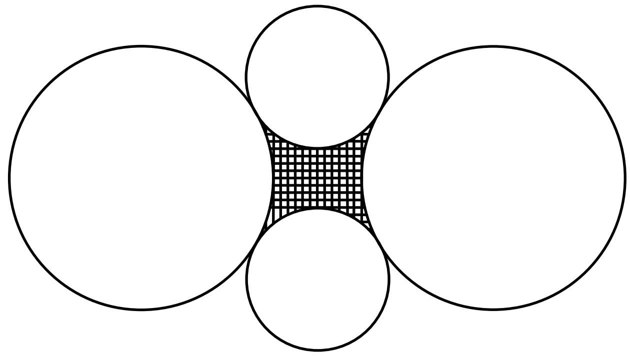
6. Let $S(n)$ be the sum of the squares of the first $n$ positive integers. Find the third-smallest integer $k$ such that $S(k)$ is divisible by $k$.

7. Tanish is playing a video game. Unfortunately, two cosmic rays randomly hit his computer and cause two bit flips! If the original number in base 2 was 110101110, and the positive difference between the original number and new number (in base 10) is 56, what is the new number (in base 10)?

   **Note:** In base 2, a *bit flip* is described by the change of a digit from 0 to 1, or 1 to 0.

8. A parabola $y = x^2/4$ is graphed on the Cartesian plane. A line $y = 2x + n$ intersects the parabola at $(a, b)$ and $(c, d)$ such that $bd = 81$. Find the product of $a$, $c$, and $-n$.

9. Two circles with radius 9 have centers located 24 units apart. Two additional circles with radius $8\sqrt{3}-9$ are drawn externally tangent to the two big circles. The area that lies between the four circles can be expressed as $a\sqrt{3}+(b\sqrt{3}-c)\pi$, where $a$, $b$, and $c$ are positive integers. Find $a+b+c$.



10. Let $ABCD$ be a square with side length 12. A circle with radius 5 is drawn inside of the square such that it is tangent to $\overline{AB}$ and $\overline{BC}$. A line passing through $D$ and tangent to the circle intersects $\overline{BC}$ at $X$. The area of $\triangle CDX$ can be expressed as $p-q\sqrt{r}$, where $p$, $q$, and $r$ are positive integers and $r$ is not divisible by any perfect square. Find $p+q+r$.

11. There are 4 people named Eric in the ABMC problem writing committee. The "Eric Test" is commonly used to decide what problems are going to be used on a test. It is defined as follows:

- The four Erics sit in a line.
- The problem is handed to the leftmost Eric. He has a $\frac{1}{2}$ probability of approving the problem.
- For the next three Erics, the probability they approve the problem depends on the number of approvals of the Erics to the left of them. Each Eric has a probability of $\frac{1}{2}-\frac{1}{2a+2}$ of approving the problem, where $a$ is the number of approvals that have come in the Erics to the left of the current Eric.
- A problem is put onto the test if at least 3 Erics approve of the problem.

For any problem that undergoes the "Eric Test", the probability that it is put onto the test is $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p+q$.

12. Let $p, q, r$ be the roots of the cubic polynomial $2x^3-19x^2+59x-60$. A rectangular prism is constructed with $p, q, r$ as side lengths (in centimeters). Let S be the set of points at most 1 cm away from some point on the prism's surface. If S encompasses a region of volume $\frac{a}{b}\pi+c$ cm$^3$, where $a$, $b$ and $c$ are integers where $a$ and $b$ are relatively prime, find $a+b+c$.

13. In a room with 4 people, one person knows a secret message. On the first day, that person tells someone else the secret message. On every succeeding day, each person who knows the secret message randomly picks another person to tell (that person may or may not know the secret message already). The expected number of days until everyone knows the secret message can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime. Find $a+b$.

14. Let $f(p)$ and $g(p)$ be defined for prime numbers $p$ as $f(p)=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{p}$, and let $g(p)=\left(((p-1)!)^p f(p)+\frac{p+1}{p}\right) \pmod{p}$. Compute $g(2)+g(3)+g(5)+\cdots+g(89)+g(97)$.

15. A red square pyramid has edges of length $2\sqrt{2}$ units each. A congruent blue pyramid is flipped upside down, such that the apex of the blue pyramid coincides with the center of the red pyramid's base (and vice versa). Then, the blue pyramid is turned $45^\circ$ around its axis of symmetry.

    Taken together, the pyramids form a certain 3D polyhedron. Its total surface area can be expressed as $a + b\sqrt{3} - c\sqrt{6}$, where $a$, $b$, and $c$ are positive integers. Find $a + b + c$.

    A diagram is given below for visualization purposes.