Acton-Boxborough Math Competition Online Contest Solutions

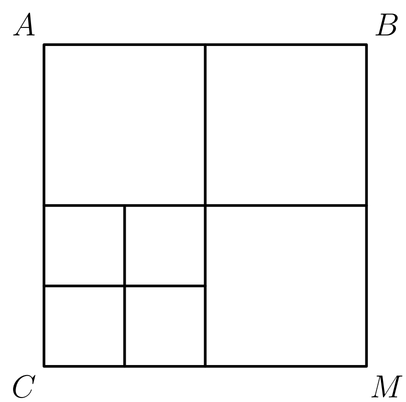
Saturday, month day — Sunday, month day, year

1. **Problem:** Evaluate $\frac{2 \cdot 0 \cdot 2 \cdot 4}{2+0+2+4}$.

   Solution: Note that the numerator is $2 \cdot 0 \cdot 2 \cdot 4 = 0$. Since 0 divided by anything is 0, the answer is $\boxed{0}$.

   Proposed by Benjamin Zhu

2. **Problem:** Square $ABMC$ is split into 4 squares of equal area. The square in the bottom-left corner is also split into 4 squares of equal area. Given that the area of the smallest squares shown in the diagram is 2, find the area of the largest square shown in the diagram.

   Solution: There are 16 total squares, each with an area of 2. Thus, the area is $2 \times 16 = \boxed{32}$.

   Proposed by Ayaan Garg and Aarush Kulkarni

3. **Problem:** Three teachers are grading a pile of math tests on Thanksgiving Day. Patrick can grade one every 3 minutes, Petra can grade one every 5 minutes, and Phil can grade one every 6 minutes.

   One day, each teacher begins grading at 9:00 AM and stops grading at 4:00 PM. How many tests are graded in total?

   Solution: There are 7 hours or 420 minutes between 9:00 AM and 4:00 PM. This means that Patrick will be able to grade $420/3 = 140$ tests, Petra will grade $420/5 = 84$ tests, and Phil will grade $420/6 = 70$ tests. Therefore, there will be $140 + 84 + 70 = \boxed{294}$ tests graded by 4:00 PM.

   Proposed by Ayaan Garg and Nathan Tan

4. **Problem:** What is the largest prime factor of $18! + 19!$?

   Note: $n! = n \cdot (n-1) \cdot ... \cdot 2 \cdot 1$.

   Solution: From the definition of a factorial, we know that $19! = 19 \cdot 18!$. Thus, we can rewrite $18! + 19!$ as $18! + 19 \cdot 18! = 18!(1+19) = 18! \cdot 20$. Expanding the factorial, we can explicitly write that as $(18 \cdot 17 \cdot 16 \cdots 3 \cdot 2 \cdot 1) \cdot 20$. The largest prime factor in this product is $\boxed{17}$.

   Proposed by Shubham Kulkarni

5. **Problem:** I have six white socks, two black socks, and eight purple socks in my drawer. On one morning, I randomly pick two socks to wear. The probability I pick two different colored socks is $\frac{p}{q}$, where p and q are relatively prime. Find $p+q$.

   Solution: There are $6 + 2 + 8 = 16$ total socks, so the number of ways to pick any two socks is $\binom{16}{2} = 120$. The number of ways to pick two white socks of the same color is $\binom{6}{2} = 15$, the number of ways to pick two black socks of the same color is $\binom{2}{2} = 1$, and the number of ways to pick two purple socks of the same color is $\binom{8}{2} = 28$.

Therefore, there are $15 + 1 + 28 = 44$ ways to pick two matching socks, and $120 - 44 = 76$ ways to pick two mismatching socks.

Finally, the probability of picking two mismatching socks is $76/120 = 19/30 \Rightarrow \boxed{49}$.

Proposed by Nathan Tan

6. **Problem:** Let $S(n)$ be the sum of the squares of the first n positive integers. Find the third-smallest integer k such that $S(k)$ is divisible by k.

   Solution: The formula for the sum of the first n squares is well-known:

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   The problem asks for the integers k such that the following value is an integer.

$$\frac{k(k+1)(2k+1)}{6k} = \frac{(k+1)(2k+1)}{6}$$

   We can discard any even k, since the numerator would be odd and therefore indivisible by 6. We can discard any k divisible by 3 for the same reason.

   Now, we just test our remaining valid values. We have $S(1) = 1$, $S(5) = 55$, and $S(7) = 140$, which are all divisible by their respective k-values. Therefore, $\boxed{7}$ is the third-smallest integer such that $S(k)$ is divisible by k.

   Proposed by Ayaan Garg

7. **Problem:** Tanish is playing a video game. Unfortunately, two cosmic rays randomly hit his computer and cause two bit flips! If the original number in base 2 was 110101110, and the positive difference between the original number and new number (in base 10) is 56, what is the new number (in base 10)?

   Note: In base 2, a *bit flip* is described by the change of a digit from 0 to 1, or 1 to 0.

   Solution: A bit flip would either add a power of two or subtract a power of two in base 10. To make a difference of 56, we have to either add 64 and subtract 8, or subtract 64 and add 8. Since 110101110 has a 0 in the 2^6's place and it has 1 in the 2^3,'s place, we have to add 56. In other words, the bit flip must have flipped the 0 in the 2^6's place to a 1 and the 1 in the 2^3's place to a 0. This means our answer is $110101110_2 + 56_{10} = 430_{10} + 56_{10} = \boxed{486}$.

   Proposed by Nathan Tan

8. **Problem:** A parabola $y = x^2/4$ is graphed on the Cartesian plane. A line $y = 2x + n$ intersects the parabola at (a, b) and (c, d) such that $bd = 81$. Find the product of a, c, and $-n$.

   Solution: Let $\mathcal{P}$ describe the parabola and ℓ describe the line. Since we need to find intersections between $\mathcal{P}$ and ℓ, we can set the y values equal to each other, giving us the equation:

$$2x + n = \frac{x^2}{4}$$
$$x^2 - 8x - 4n = 0.$$

   The roots of this equation are the values of x where $\mathcal{P}$ intersects ℓ, which are a and c. Vieta's Formula states that the product of the roots of a polynomial in the form $p_2x^2 + p_1x + p_0c$ is equal to p_0, the constant term. Thus, $ac = -4n$. Multiplying again by $-n$, our final answer in terms of n is $4n^2$.

To find n^2, we use the equation $y = x^2/4$ twice. Setting y as b or d would make $x = a$ or $x = c$ respectively, so we have $b = a^2/4$ and $d = c^2/4$. Multiplying these two equations together, we have

$$bd = \frac{(ac)^2}{4^2}.$$

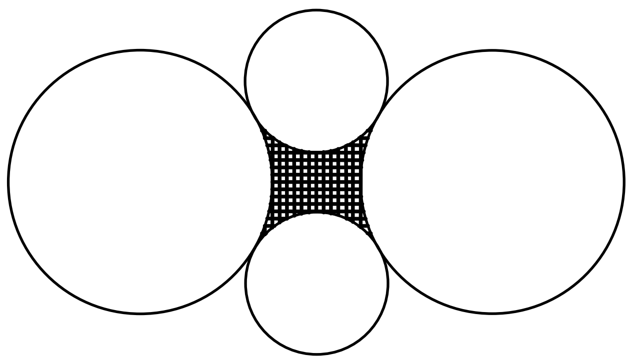
We know $bd = 81$, and we already found $ac = -4n$. Plugging these two values in, we have

$$81 = \frac{(-4n)^2}{16} \Rightarrow 81 = n^2.$$

Thus, our answer is $4(81) = \boxed{324}$.

Proposed by Daniel Ren

9. **Problem:** Two circles with radius 9 have centers located 24 units apart. Two additional circles with radius $8\sqrt{3} - 9$ are drawn externally tangent to the two big circles. The area that lies between the four circles can be expressed as $a\sqrt{3} + (b\sqrt{3} - c)\pi$, where a, b, and c are positive integers. Find $a + b + c$.



Solution: Let the centers of the two large circles be W and Y, while the centers of the two smaller ones are X and Z. From the radii given, we know that $WX = XY = 9 + 8\sqrt{3} - 9 = 8\sqrt{3}$. We also know that $WY = 24$. So, we have an isosceles triangle $\triangle WXY$ with sides of length $8\sqrt{3}, 8\sqrt{3}, 24$.

Drawing the altitude to the side of length 24 gives us two congruent right triangles with hypotenuse of length $8\sqrt{3}$ and a leg of length 12. Using the Pythagorean Theorem, we can find the other leg to have a length of $4\sqrt{3}$, which tells us that this is a 30-60-90 triangle. So, we know that $\angle WXY = \angle WZY = 120^\circ$ (as they are formed by the two 60° angles) This also means that $\angle XWZ = \angle XYZ = 60^\circ$ (formed by the two 30° angles).

Now, we see that the shaded area can be expressed as the area of the rhombus $WXYZ$ with side length $8\sqrt{3}$ minus the area of the four sectors of the four circles bounded by the rhombus. Specifically, there are two sectors with radius 9 and angle 60°, and two sectors with radius $8\sqrt{3} - 9$ and angle 120°.

We know one diagonal, WY, of the rhombus has length 24, and we can find the length of XZ, the other diagonal, since we know $\angle XWZ = 60^\circ$ and thus $\triangle XWZ$ is equilateral. This gives us $XZ = 8\sqrt{3}$. Therefore, the area of the rhombus is

$$\frac{24 \cdot 8\sqrt{3}}{2} = 96\sqrt{3}.$$

Now, to find the areas of the sectors. The sectors with radius 9 and angle 60° have areas of

$$\frac{9^2 \cdot \pi}{6} = \frac{27\pi}{2},$$

so together they have an area of 27π.

Similarly, the sectors with radius $8\sqrt{3}-9$ and angle $120°$ have areas of

$$\frac{(8\sqrt{3}-9)^2\pi}{3} = \frac{(273-144\sqrt{3})\pi}{3} = (91-48\sqrt{3})\pi,$$

so together they have an area of $(182-96\sqrt{3})\pi$.

Putting it all together, we find that the area of the shaded region is

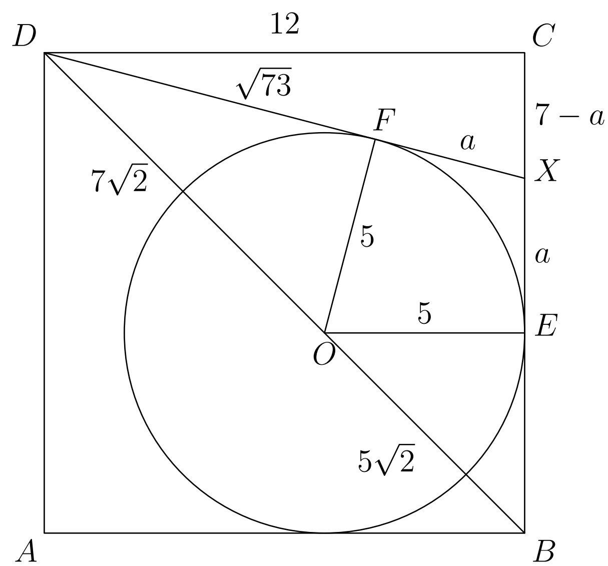
$$\begin{aligned}&96\sqrt{3}-27\pi-182\pi+96\pi\sqrt{3}\\&96\sqrt{3}+96\pi\sqrt{3}-209\pi\\&96\sqrt{3}+(96\sqrt{3}-209)\pi.\end{aligned}$$

Our final answer is $96+96+209=\boxed{401}$.

Proposed by Tanish Parida

10. **Problem:** Let $ABCD$ be a square with side length 12. A circle with radius 5 is drawn inside of the square such that it is tangent to $\overline{AB}$ and $\overline{BC}$. A line passing through D and tangent to the circle intersects $\overline{BC}$ at X. The area of $\triangle CDX$ can be expressed as $p-q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by any perfect square. Find $p+q+r$.

Solution: A diagram is shown below. Let O be the center of the circle, E be the point of tangency on $\overline{BC}$, and F be the point of tangency on $\overline{DX}$.



By equal tangents to a circle, $EX = FX$. Let $EX = FX = a$ for some number a. Since $BE = 5$, $EC = BC - BE = 12 - 5 = 7$. Then, $XC = EC - EX = 7 - a$. Since $\overline{BD}$ is the diagonal of $ABCD$, $BD = 12\sqrt{2}$, and because $BO = 5\sqrt{2}$, $DO = BD - BO = 7\sqrt{2}$.

$\overline{OF}$ is perpendicular to $\overline{DX}$, so $\triangle OFD$ is right at $\angle F$. By the Pythagorean theorem

$$\begin{aligned}FD^2 &= OD^2 - OF^2\\&= (7\sqrt{2})^2 - 5^2\\&= 98 - 25 = 73\\FD &= \sqrt{73}.\end{aligned}$$

Now, since $\triangle CDX$ is right at $\angle C$, and $DX = XF + FD = a + \sqrt{73}$, by the Pythagorean theorem

$$\begin{aligned} DX^2 &= DC^2 + CX^2 \\ (a+\sqrt{73})^2 &= 12^2 + (7-a)^2 \\ a^2 + 2a\sqrt{73} + 73 &= 144 + 49 - 14a + a^2 \\ 14a + 2a\sqrt{73} &= 120 \\ (7+\sqrt{73})a &= 60. \end{aligned}$$

We solve for the simplified version of a:

$$a = \frac{60}{7+\sqrt{73}} = \frac{60(7-\sqrt{73})}{49-73} = \frac{420-60\sqrt{73}}{-24} = \frac{5\sqrt{73}-35}{2}.$$

Therefore, we have

$$CX = 7 - a = 7 - \frac{5\sqrt{73}-35}{2} = \frac{49-5\sqrt{73}}{2}.$$

So the area of $\triangle CDX$ is

$$[\triangle CDX] = \frac{1}{2}(CD)(CX) = \frac{1}{2}(12)\left(\frac{49-5\sqrt{73}}{2}\right) = 147 - 15\sqrt{73},$$

and our final answer is $147 + 15 + 73 = \boxed{235}$.

Proposed by Daniel Ren

11. **Problem:** There are 4 people named Eric in the ABMC problem writing committee. The "Eric Test" is commonly used to decide what problems are going to be used on a test. It is defined as follows:

- The four Erics sit in a line.
- The problem is handed to the leftmost Eric. He has a $\frac{1}{2}$ probability of approving the problem.
- For the next three Erics, the probability they approve the problem depends on the number of approvals of the Erics to the left of them. Each Eric has a probability of $\frac{1}{2} - \frac{1}{2a+2}$ of approving the problem, where a is the number of approvals that have come in the Erics to the left of the current Eric.
- A problem is put onto the test if at least 3 Erics approve of the problem.

For any problem that undergoes the "Eric Test", the probability that it is put onto the test is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution: Note that if the first Eric denies the problem, Eric 2 has a $\frac{1}{2} - \frac{1}{2} = 0$ probability of approving it. So, Eric 1 must approve the problem in order for it to be put on the test. So, we need to consider three cases:

- **Case I:** Eric's 1, 2, and 3 approve.
- **Case II:** Eric's 1, 2, and 4 approve.
- **Case III:** Eric's 1, 3, and 4 approve.

Case I: Eric 1's probability of approving is $\frac{1}{2}$. After Eric 1 approves it, Eric 2 has a $\frac{1}{2} - \frac{1}{2\cdot 1+2} = \frac{1}{4}$ chance of approving. Finally, Eric 3 has a $\frac{1}{2} - \frac{1}{2\cdot 2+2} = \frac{1}{3}$ chance of approving. It does not matter whether or not Eric 4 approves, as 3 Erics have already approved. Therefore, the probability for this case is

$$\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{3} = \frac{1}{24}.$$

Case II: Again, Eric 1 has a $\frac{1}{2}$ chance of approving, and Eric 2 has a $\frac{1}{4}$ chance of approving. The probability that Eric 3 rejects the problem is $1 - \frac{1}{3} = \frac{2}{3}$. Now, the probability that Eric 4 approves is $\frac{1}{2} - \frac{1}{2 \cdot 2 + 2} = \frac{1}{3}$. Therefore, the probability for this case is

$$\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{36}.$$

Case III: Eric 1 has a $\frac{1}{2}$ chance of approving. Eric 2 has a $1 - \frac{1}{4} = \frac{3}{4}$ chance of rejecting the problem. Since there is only 1 approval, Eric 3 has the same chance of approving as Eric 2 did, which is $\frac{1}{4}$. Finally, Eric 4 has a $\frac{1}{3}$ of approving. So, the probability for this case is

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{32}.$$

Combining all 3 cases, the probability of at least 3 Eric's approving of a problem is $\frac{1}{24} + \frac{1}{36} + \frac{1}{32} = \frac{29}{288}$. Therefore, $p + q = 29 + 288 = \boxed{317}$.

Proposed by Tanish Parida

12. **Problem:** Let p, q, r be the roots of the cubic polynomial $2x^3 - 19x^2 + 59x - 60$. A rectangular prism is constructed with p, q, r as side lengths (in centimeters). Let S be the set of points at most 1 cm away from some point on the prism's surface. If S encompasses a region of volume $\frac{a}{b}\pi + c$ cm^3, where a, b and c are integers where a and b are relatively prime, find $a + b + c$.

Solution: There are two cases to consider. First, there are the points that are inside the rectangular prism, and second, there are points that are outside of the rectangular prism.

Case I: Points that are inside the rectangular prism.

We can use complementary counting to calculate the volume of the first case. The points that are more than 1 cm away from the faces must be in a rectangular prism of side lengths $p - 2$, $q - 2$, and $r - 2$. This is because the volume of the points not included is 1 cm away from each of the faces on all sides.

Therefore, the points within 1 cm of the faces lie in a volume of

$$pqr - (p-2)(q-2)(r-2).$$

This simplifies to

$$pqr - [pqr - 2(pr + pq + qr) + 4(p + q + r) - 8]$$
$$2(pr + pq + qr) - 4(p + q + r) + 8.$$

By Vieta's formulas on $2x^3 - 19x^2 + 59x - 60$,

$$pr + pq + qr = \frac{59}{2} \quad (1)$$
$$p + q + r = \frac{19}{2} \quad (2)$$

Plugging (1) and (2) into our equation yields

$$2 \cdot \frac{59}{2} - 4 \cdot \frac{19}{2} + 8 = 29.$$

Therefore, the set of points inside the rectangular prism that satisfy the given condition has volume 29 cm^3.

Case II: Points that are outside of the rectangular prism.

The set of points exterior to the prism within 1 cm of it can be described in 3 categories:

1. Points in the rectangular prisms of height 1 directly on top of each face of the original rectangular prism
2. Points that are on the quarter-cylinders of radius 1 for each of the edges of the original rectangular prism (These points make up the area between pairs of rectangular prisms of height 1)
3. Points that are on the one-eighth spheres of radius 1 for each of the corners of the original rectangular prism (These points make up the area between triplets of the quarter-cylinders)

For points in the 1st category, the volume of the rectangular prism added on top of the faces of the original rectangular prism has the same value as the area of the face (with the height of each prism being 1). There are 6 faces, with their areas being pq, pq, pr, pr, qr, and qr. Therefore, the volume of the points in the 1st category is $2(pq + pr + qr)$.

For points in the 2nd category, the quarter cylinders have base area $\frac{1}{4}\pi$, since the radius of each is 1. The height is either p, q, or r. Summing over all 12 quarter-cylinders, we get that the volume of the points in this category is $4 \cdot \frac{p}{4}\pi + 4 \cdot \frac{q}{4}\pi + 4 \cdot \frac{r}{4}\pi = (p + q + r)\pi$.

For points in the 3rd category, all the one-eighth spheres have radius 1, so combining them gives us a sphere of radius 1. Therefore, the volumes of the points in this category is $\frac{4}{3}\pi$.

Combining all categories in Case II yields

$$2(pq + pr + qr) + (p + q + r)\pi + \frac{4}{3}\pi.$$

Plugging in (1) and (2) makes this $59 + \frac{19}{2}\pi + \frac{4}{3}\pi$ cm^3.

Combining the cases, the total volume in cm^3 is

$$29 + 59 + \frac{19}{2}\pi + \frac{4}{3}\pi.$$

This simplifies to

$$\frac{65}{6}\pi + 88.$$

Therefore, $a + b + c = 65 + 6 + 88 = \boxed{159}$.

Proposed by Eric Xiang

13. **Problem:** In a room with 4 people, one person knows a secret message. On the first day, that person tells someone else the secret message. On every succeeding day, each person who knows the secret message randomly picks another person to tell (that person may or may not know the secret message already). The expected number of days until everyone knows the secret message can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Find $a + b$.

Solution: We can think about this problem as a process involving 4 different states. There are 4 "states," one for 1 person knowing the secret, one for 2 people knowing the secret, and so on. Each day, we can move from one state to another as the secret spreads.

First, let's find the probability that it moves from one state to the next. Starting with only one person knowing the secret, it moves to the state with two people knowing the secret with probability 1, as the person with the secret must tell someone else the secret.

From two people knowing the secret, we note that there are three possible states that it can move to:

1. **It stays the same.** This happens if the two people each tell each other the secret. Since each person has 3 options for who to tell the secret to, this happens with probability $(\frac{1}{3})^2 = \frac{1}{9}$.
2. **It moves to the state where four people know.** This happens where each person who knows tells a different person who does not know. The first person can tell someone who doesn't know with probability $\frac{2}{3}$, while the second person must tell the remaining person with probability $\frac{1}{3}$, giving a total probability of $(\frac{2}{3})(\frac{1}{3}) = \frac{2}{9}$.

3. **It moves to the state where three people know.** Since there are only three options, and we know the probability that it moves to the other two, we can simply calculate this as $1 - \frac{1}{9} - \frac{2}{9} = \frac{2}{3}$
   Note: We can also calculate case 3 by doing casework on if the two people tell the same person who doesn't know, or if one person tells the other person that knows, and then the other person tells someone who doesn't know.

Starting with three people knowing, it can stay at three people with probability $(\frac{2}{3})^3 = \frac{8}{27}$ (each person tells someone else who knows with probability $\frac{2}{3}$), and it can also move to four people knowing with probability $1 - \frac{8}{27} = \frac{19}{27}$ (since there are only two options for which state we can go next).

Now, we are ready to solve for the expected value. Let e_1 be the expected value of days it will take starting with one person knowing the secret. Similarly, let e_2, e_3, and e_4 be the expected value of days it will take starting with two, three, and four people knowing. Now, we can write these expected values in terms of each other. For example, we can write

$$e_1 = e_2 + 1.$$

This is because e_1 will always become e_2 after one day, so we can simply add 1 to e_2 to get e_1. We also have

$$e_2 = \frac{1}{9}e_2 + \frac{2}{3}e_3 + \frac{2}{9}e_4 + 1.$$

This is because e_2 goes to the state of two people knowing with probability $\frac{1}{9}$, three people knowing with $\frac{2}{3}$, four people knowing with $\frac{2}{9}$, and moving to each one requires one additional day, hence the $+1$ at the end. Similarly, we have

$$e_3 = \frac{8}{27}e_3 + \frac{19}{27}e_4 + 1,$$

and finally, we have

$$e_4 = 0,$$

since if we start with four people knowing, the condition is always satisfied so it takes 0 days. So, we have the equations

$$\begin{aligned} e_1 &= e_2 + 1 \\ e_2 &= \frac{1}{9}e_2 + \frac{2}{3}e_3 + \frac{2}{9}e_4 + 1 \\ e_3 &= \frac{8}{27}e_3 + \frac{19}{27}e_4 + 1 \\ e_4 &= 0. \end{aligned}$$

We plug in e_4 into the equation for e_3 to get

$$e_3 = \frac{8}{27}e_3 + 0 + 1 \Rightarrow \frac{19}{27}e_3 = 1 \Rightarrow e_3 = \frac{27}{19}.$$

Plugging this into the equation for e_2, we get

$$e_2 = \frac{1}{9}e_2 + \frac{2}{3} \cdot \frac{27}{19} + 0 + 1 \Rightarrow e_2 = \frac{333}{152}.$$

Finally, we have

$$e_1 = e_2 + 1 = \frac{333}{152} + 1 = \frac{485}{152}.$$

Our final answer is $485 + 152 = \boxed{637}$.

Proposed by Tanish Parida

14. **Problem:** Let $f(p)$ and $g(p)$ be defined for prime numbers p as $f(p) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{p}$, and let $g(p) = \left(((p-1)!)^p f(p) + \frac{p+1}{p}\right) \pmod{p}$. Compute $g(2) + g(3) + g(5) + \cdots + g(89) + g(97)$.

Solution: By *Wilson's Theorem*, for a prime p:

$$(p-1)! \equiv -1 \pmod{p}.$$

Then raising both sides to the power of p, we get:

$$((p-1)!)^{p-1} \equiv (-1)^{p-1} \pmod{p}.$$

Notice that we left out one of the factors of $(p-1)!$, this is intentional! If we multiply out

$$(p-1)!f(p) = \sum_{k=1}^{p-1} \frac{(p-1)!}{k} - 1/p,$$

where the $-\frac{1}{p}$ term comes from using the fact that $(p-1)! \equiv -1 \pmod{p}$.

Now notice that if $\frac{(p-1)!}{i} \equiv \frac{(p-1)!}{j} \pmod{p}$ for some $1 \leq i, j \leq p-1$, we must have $i \cdot \frac{(p-1)!}{ij} \equiv j \cdot \frac{(p-1)!}{ij} \pmod{p}$, so since p is prime we can multiply both sides by the inverse of $\frac{(p-1)!}{ij}$ to get $i = j$.

Thus all terms of the sum $\sum_{k=1}^{p-1} \frac{(p-1)!}{k}$ are distinct modulo p. Obviously, none of these terms are divisible by p, meaning that $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 1 + 2 + \cdots + (p-1) \pmod{p}$, but $1 + 2 + \cdots + (p-1) = \frac{p(p-1)}{2} \equiv 0 \pmod{p}$ for all odd primes, so we simply have

$$g(p) = -\frac{1}{p}((p-1)!)^{p-1} + \frac{p+1}{p}.$$

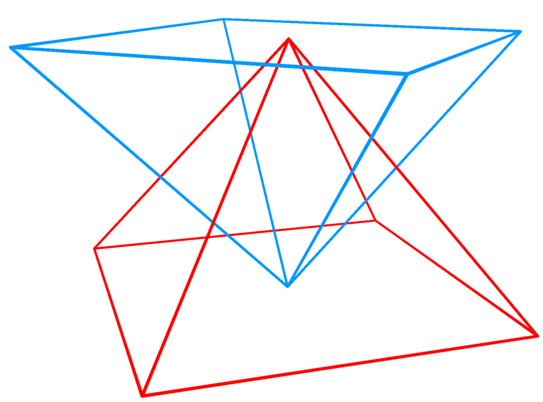
When $p = 2$, we can compute $g(p) = 1$ by hand, and when $p \geq 3$, we have $((p-1)!)^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod{p}$, so $g(p) = -\frac{1}{p} + \frac{p+1}{p} = 1$. Thus the sum $g(2) + g(3) + g(5) + \cdots + g(89) + g(97)$ is simply equal to the number of prime numbers less than 100, and counting them up we find that the answer is $\boxed{25}$.

Proposed by Bryan Li

15. **Problem:** A red square pyramid has edges of length $2\sqrt{2}$ units each. A congruent blue pyramid is flipped upside down, such that the apex of the blue pyramid coincides with the center of the red pyramid's base (and vice versa). Then, the blue pyramid is turned $45°$ around its axis of symmetry.

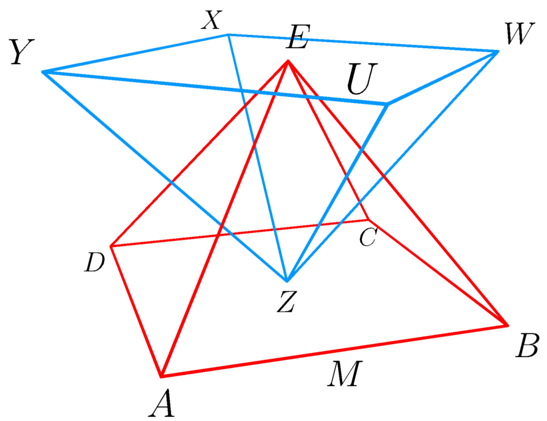
Taken together, the pyramids form a certain 3D polyhedron. Its total surface area can be expressed as $a + b\sqrt{3} - c\sqrt{6}$, where a, b, and c are positive integers. Find $a + b + c$.

A diagram is given below for visualization purposes.

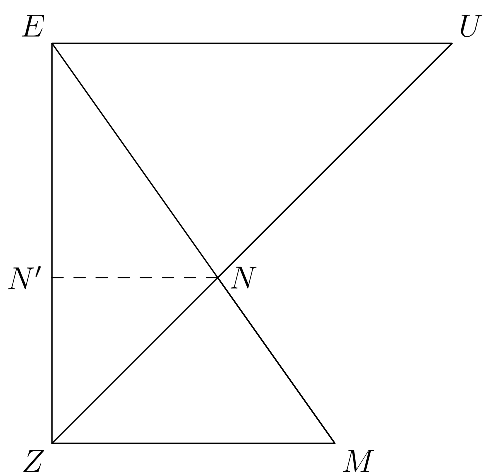


Solution: Let A, B, C, and D be the vertices of the square base of the red pyramid, in that order. Let the apex of the red pyramid by E and the apex of the blue pyramid be Z. Let U, W, X and Y be the vertices of the blue pyramid such that $\overline{UZ}$ intersects $\triangle ABE$, $\overline{WZ}$ intersects $\triangle BCE$, $\overline{XZ}$ intersects $\triangle CDE$, and $\overline{YZ}$ intersects $\triangle DAE$. Finally, let M be the midpoint of AB, N be the intersection of $\triangle ABE$ and $\overline{UZ}$, O be the intersection of $\triangle YUZ$ and $\overline{AE}$, and P be the intersection of $\triangle UWZ$ and BE.

The total surface area of the polyhedron is 8 times the area of $AONPB$ plus the area of the two square bases.



Consider the plane containing E, U, M and Z. Note that $\overline{EM}$ intersects $\overline{UZ}$ at N. Let the foot of the perpendicular from N to EZ be N'.



Notice that $EU = 2$, $EZ = 2$, $ZM = \sqrt{2}$, $\triangle EZM$ is a right triangle, and that $\triangle UEZ$ is an isosceles right triangle. We see that $\frac{1}{NN'} = \frac{1}{EM} + \frac{1}{FQ}$ (this can be proven through similar triangles). Therefore,

$$\frac{1}{NN'} = \frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{2}$$

and so

$$NN' = \frac{2}{\sqrt{2}+1} = \frac{2}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = 2\sqrt{2}-2.$$

Since $\triangle NN'Z$ must be an isosceles right triangle, we note that

$$ZN = NN' \cdot \sqrt{2} = (2\sqrt{2}-2)\sqrt{2} = 4 - 2\sqrt{2}.$$

It is trivially easy find that $ZU = 2\sqrt{2}$, therefore we see that

$$NU = ZU - ZN = 2\sqrt{2} - (4 - 2\sqrt{2}) = 4\sqrt{2} - 4.$$

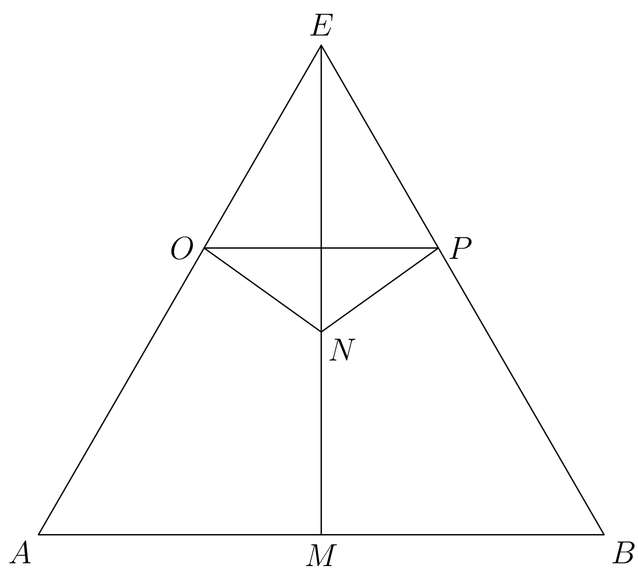
Additionally, note that $EM = \sqrt{6}$ from the Pythagorean Theorem. Using similar triangles $\triangle ENN' \sim \triangle EMZ$, we can determine that

$$\frac{EI}{NN'} = \frac{EM}{ZM}.$$

Plugging in the known lengths, we know that

$$\frac{EN}{2\sqrt{2}-2} = \frac{\sqrt{6}}{\sqrt{2}} \Rightarrow EN = 2\sqrt{6} - 2\sqrt{3}.$$

Now, consider $\triangle ABE$.



Notice that $AO = NU$ due to the congruent pyramids, so we must have

$$AO = 4\sqrt{2} - 4.$$

Since $\triangle ABE$ is an equilateral triangle with side length $2\sqrt{2}$, note that

$$OE = AE - AO = 2\sqrt{2} - (4\sqrt{2} - 4) = 4 - 2\sqrt{2}.$$

We see that $\triangle OEP$ is also a right triangle, so $OE = OP = 4 - 2\sqrt{2}$. Note that quadrilateral $OEPN$ is a kite, so the area of quadrilateral $OEPN$ is

$$\frac{1}{2} \cdot EN \cdot OP = \frac{1}{2} \cdot (2\sqrt{6} - 2\sqrt{3}) \cdot (4 - 2\sqrt{2}) = 6\sqrt{6} - 8\sqrt{3}.$$

The area of ABE is $\frac{(2\sqrt{2})^2\sqrt{3}}{4} = 2\sqrt{3}$, so we see that

$$[AONPB] = [ABE] - [OEPN] = 2\sqrt{3} - (6\sqrt{6} - 8\sqrt{3}) = 10\sqrt{3} - 6\sqrt{6}.$$

The area of $ABCD$ is $(2\sqrt{2})^2 = 8$, so the total surface area of the polyhedron is

$$8(10\sqrt{3} - 6\sqrt{6}) + 2(8) = 16 + 80\sqrt{3} - 48\sqrt{6}.$$

We get $a = 16, b = 80$, and $c = 48$, so $a + b + c = 16 + 80 + 48 = \boxed{144}$.

Proposed by Tanish Parida and Eric Li