# Acton-Boxborough Math Competition Online Contest

Saturday, October 19 — Sunday, October 20, 2024

# Contest Rules and Format

The 2024 October Contest consists of 15 problems worth 1 point each — each with an answer between 0 and 100,000. The contest window is

Saturday, October 19 to Sunday, October 20.

The following rules supersede any others regarding the Online Contests.

- To submit your responses, fill out a form on our website, `abmathcompetition.wordpress.com`. Include your email, name, school, grade (we require this information for prize purposes).
- Submissions will close at 10:00PM Eastern Standard Time, on the final day of the round.
- Computational aids such as WolframAlpha, calculators, abaci, and other related materials are prohibited. Graphing technology such as GeoGebra, Desmos, or a graphing calculator are also prohibited. Drawing aids such as rulers and protractors are permissible.
- You may only work individually on the problems—consultation with others is not permitted.
- You may take as much time as you wish during the contest window.
- Ties will be broken by the most difficult problem solved, determined by the number of correct answers to the question. If problem A is solved by $a$ contestants, and problem B is solved by $b$ contestants, with $a < b$, then problem A is said to be "more difficult" than B.

# Awards and Prizes

- Top scoring contestants within each round will have their names posted on our website.
- Scores will be emailed to contestants within one week of the contest's end.
- Top scoring contestants across all three rounds will receive prizes.
- Physical prizes (such as AoPS gift certificates, merchandise, calculators, etc.) will be given to top scorers if they attend the ABMC onsite contest.

# Thanks to our Sponsors!

Good luck!

# Problems

1. 20 years ago, my father was 4 times my age. Today, he is twice my age. How old am I today?

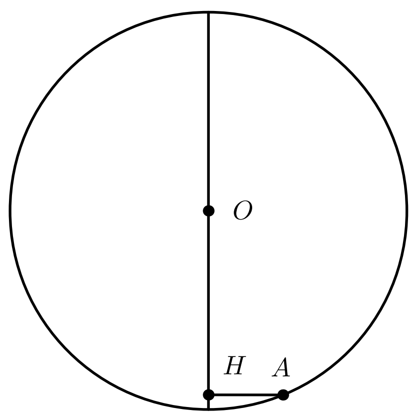
2. Omar rolls 2 dice. Let $\frac{a}{b}$ be the probability that the sum of the two numbers is a prime number, where $a$ and $b$ are relatively prime. Find $a + b$.

3. Hillary has a disk of radius 6 and one sphere of radius $n$. If the value of the surface area of the sphere is equal to the area of the disk, then find the value $n$.

4. Find the sum of all positive integers $n$ such that $n$ is a divisor of $n^2 + 5$.

5. Aiden is running on a soccer field and he suddenly passes out! Luckily, he is in the middle of the field in the center circle such that anyone can run over him. He lies perpendicular to the middle line such that his toes touch the middle line and his head touches the circle. Given that the circle has a diameter of 60 feet, Aiden is 6 feet tall, and the distance between his toes to the center of the field can be written as $\sqrt{b}$ feet, find $b$.

   **Note:** $\overline{AH}$ represents his head and toes, respectively, point $O$ represents the center of the circle, and the diagram is not to scale.




6. A *funky* date is a date where, when written in the form MM/DD/YYYY, all of the nonzero digits are the same. Today is February 2nd, 2022. Find the number of days between the *funky* dates directly before and directly after this date.

7. How many pairs of positive integers $(a, b)$ for $a < b$ satisfy $\frac{1}{a} + \frac{3}{b} = \frac{1}{2024}$?

8. For a triplet of positive integers $(a, b, c)$; we have $\gcd(a, b) = 12$, $\gcd(b, c) = 12$, $\gcd(c, a) = 120$; and $\text{lcm}(a, b) = 1680$, $\text{lcm}(b, c) = 2520$, $\text{lcm}(c, a) = 720$. Find $a + b + c$.

9. A caterpillar wants to go from $(0, 0)$ to $(5, 5)$ on the coordinate plane. It can only move up and right. If it reaches $(3, 3)$ at any point in time, it teleports to $(1, 1)$. Determine how many ways the caterpillar can reach its destination within 20 moves.

10. Circles $\omega_1$ and $\omega_2$ have radii 6 and 8, respectively, with the distance between their centers being 10. The two circles intersect at points $A$ and $B$. A line through $A$ intersects the two circles at points $X$ and $Y$. What is the maximum length of $XY$?

11. There are four identical spheres, each with a radius of 1, positioned in space such that every sphere touches the other three externally (their surfaces are tangent at exactly one point for each pair). Now, a fifth sphere is placed among them such that it touches all four of these spheres externally as well. The diameter of the fifth sphere can be written as $\sqrt{a} - b$ for positive integers $a$ and $b$. What is $a + b$?

12. The cubic $x^3 + 8x^2 + 5x + 3$ has roots $r$, $s$, and $t$. There exists a unique set of numbers $A$, $B$, and $C$, such that $x^3 + Ax^2 + Bx + C$ has $r + s$ as a root. What is value of $(A + B + C)^2$?

13. Let $f(x)$ be the sum of the factors of $x$. Given that $b$ is the smallest positive integer with $6^6$ factors, find the largest prime factor of $f(b)$.

14. April, Mei, June, and Jason are playing a game. They take turns playing the game in this order. April begins by rolling a 12 sided dice with the names of all 12 calendar months on it; she wins if she rolls the side with her name on it, and otherwise the game continues and she passes it to the next person. Similarly, Mei (May) and then June do the same. On Jason's turn, he wins if the dice lands on any of the months July, August, September, October, and November. The probability of Jason being the first winner is $\frac{m}{n}$, where $m$ and $n$ are relatively prime. Find $m+n$.

15. Let $a_1, a_2, a_3$, and $a_4$ be roots of the polynomial $Mx^4 + Nx^2 + M$. Given that $M$ and $N$ are both positive integers, and

$$\prod_{i=1}^{4}(8-16a_i)^2\left(24-\frac{16}{a_i}\right) = 2374311,$$

find $M+N$.

**Note:** $2374311 = 3 \cdot 71^2 \cdot 157$.