Acton-Boxborough Math Competition Online Contest Solutions

Saturday, November 19 — Sunday, November 20, 2022

1. **Problem:** Calculate $A \cdot B + M \cdot C$, where $A = 1, B = 2, C = 3, M = 13$.

   Solution: We substitute the values into the expression: $1 \cdot 2 + 13 \cdot 3 = 2 + 39 = \boxed{41}$.

   Proposed by Daniel Cai

2. **Problem:** What is the remainder of $\frac{2022 \cdot 2023}{10}$?

   Solution: The remainder when a number is divided by ten is the units digit of the number. The units digit of the product of two numbers is also only affected by the units digits of the numbers. So, the desired remainder is the units digit of $2 \cdot 3 = \boxed{6}$.

   Proposed by Ivy Shi

3. **Problem:** Daniel and Bryan are rolling fair 7-sided dice. If the probability that the sum of the numbers that Daniel and Bryan roll is greater than 11 can be represented as the fraction $\frac{a}{b}$ where a, b are relatively prime positive integers, what is $a + b$?

   Solution: We can simply list out the ways that the sum is greater than 11:

   $(5, 7), (6, 6), (7, 5), (6, 7), (7, 6), (7, 7)$, for a total of 6 ways. There are $7 \cdot 7 = 49$ total possibilities, so the probability is $\frac{6}{49}$, which gives $a + b = 6 + 49 = \boxed{55}$.

   Proposed by Raymond Gao

4. **Problem:** Billy can swim the breaststroke at 25 meters per minute, the butterfly at 30 meters per minute, and the front crawl at 40 meters per minute. One day, he swam without stopping or slowing down, swimming 1130 meters. If he swam the butterfly for twice as long as the breaststroke, plus one additional minute, and the front crawl for three times as long as the butterfly, minus eight minutes, for how many minutes did he swim?

   Solution: Let a represent the number of minutes Billy swam the breaststroke, b represent the number of minutes Billy swam the butterfly, and c represent the number of minutes Billy swam the front crawl. Converting the given information into equations, we have $b = 2a + 1$ and $c = 3b - 8$. Substituting the first equation into the second and simplifying gives $c = 6a - 5$. Since he swam a total of 1130 meters, we know that $25a + 30b + 40c = 1130$. Dividing both sides by 5 gives $5a + 6b + 8c = 226$. Substituting in our expressions for b and c gives $5a + 6(2a + 1) + 8(6a - 5) = 226$. Simplifying, we have $65a + 6 - 40 = 226$, or $65a = 260$, giving $a = 4$. We wish to find the value of $a + b + c$. Using our expressions for b and c, we have $a + b + c = a + 2a + 1 + 6a - 5 = 9a - 4$. Plugging in our result for a gives an answer of $4 \cdot 9 - 4 = 36 - 4 = \boxed{32}$ minutes.

   Proposed by Ryon Das

5. **Problem:** Elon Musk is walking around the circumference of Mars trying to find aliens. If the radius of Mars is exactly 3396.2 km, and Elon Musk is 73 inches tall, the difference in distance traveled between the top of his head and the bottom of his feet in inches can be expressed as $a\pi$ for an integer a. Find a. (1 yard is exactly 0.9144 meters).

   Solution: The distance traveled by the bottom of Elon's feet and the top of his head can be represented by two concentric circles with differing radii of 73 inches. Rather than converting Mars' radius into inches, let the radius of Mars be r inches. The smaller circle then has a radius of r, giving a circumference of $2r\pi$. The larger circle has a radius of $r + 73$, giving a circumference of $(2r + 146)\pi$. The difference is 146π, giving an answer of $a = \boxed{146}$.

   Proposed by Bryan Li

6. **Problem:** Lukas is picking balls out of his five baskets labeled 1,2,3,4,5. Each basket has 27 balls, each labeled with the number of its respective basket. What is the least amount of times Lukas must take

one ball out of a random basket to guarantee that he has chosen at least 5 balls labeled "1"? If there are no balls in a chosen basket, Lukas will choose another random basket.

Solution: We can consider the "worst" possible scenario. This occurs when Lukas chooses all the balls in the baskets labeled 2,3,4,5 before choosing 5 balls labeled 1. This gives us $27 \cdot 4 = 108$ balls already. Then, Lukas can only pick 5 balls labeled 1, giving a final answer of $108 + 5 = \boxed{113}$.

Proposed by Daniel Cai

7. **Problem:** Given $35_a = 42_b$, where positive integers a, b are bases, find the minimum possible value of the sum $a + b$ in base 10.

   Solution: In base 10, this equation is $3a + 5 = 4b + 2$. Subtracting 2 from both sides yields $3a + 3 = 4b$, or $3(a + 1) = 4b$. The least possible values of a and b occur when we have $a + 1 = 4 \Rightarrow a = 3$ and $b = 3$. However, 35 is not be a valid number in base 3, nor is 42 a valid number in base 3. Then, the next smallest solution is $a + 1 = 8 \Rightarrow a = 7$ and $b = 6$. This is a valid solution, giving an answer of $7 + 6 = \boxed{13}$.

   Proposed by Ryon Das

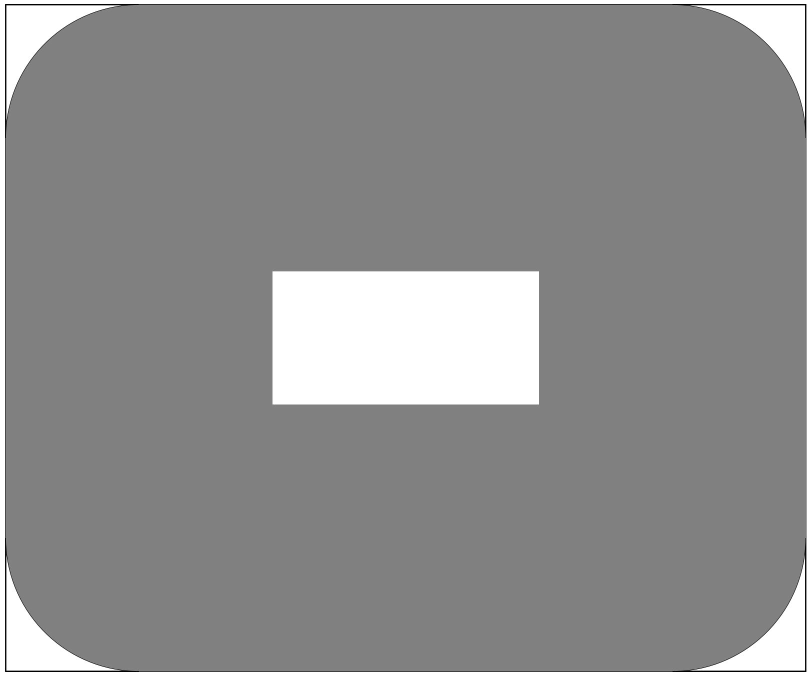
8. **Problem:** Jason is playing golf. If he misses a shot, he has a 50 percent chance of slamming his club into the ground. If a club is slammed into the ground, there is an 80 percent chance that it breaks. Jason has a 40 percent chance of hitting each shot. Given Jason must successfully hit five shots to win a prize, what is the expected number of clubs Jason will break before he wins a prize?

   Solution: Since Jason has a 40 percent chance of hitting one shot, for every shot Jason attempts he is expected to hit $\frac{2}{5}$ of a shot and miss $\frac{3}{5}$ of a shot. So, if Jason makes 5 shots, he is expected to miss $\frac{\frac{3}{5}}{\frac{2}{5}} \cdot 5 = \frac{15}{2}$ shots. Then, the expected number of clubs he will slam into the ground is $\frac{15}{2} \cdot 50\% = \frac{15}{2} \cdot \frac{1}{2} = \frac{15}{4}$. Finally, there is an $80\% = \frac{4}{5}$ chance each of these clubs will break, giving a value of $\frac{15}{4} \cdot \frac{4}{5} = \boxed{3}$.

   Solution 2: We find that for any given shot there is a $\frac{2}{5}$ chance he wins a prize, a $\frac{3}{5} \cdot \frac{1}{2} \cdot \frac{4}{5} = \frac{6}{25}$ chance he breaks the club, and a $\frac{3}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{1}{5} = \frac{9}{25}$ chance he misses, but doesn't break the club. Let e be the expected number of clubs Jason will break before he hits a shot. We know $e = \frac{6}{25}(e + 1) + \frac{9}{25}e \Rightarrow e = \frac{3}{5}e + \frac{6}{25} \Rightarrow e = \frac{3}{5}$. We want to find the expected number of clubs Jason will break before he wins his prize, or hits 5 shots. This value is $5e = \boxed{3}$.

   Proposed by Daniel Cai

9. **Problem:** Circle O with radius 1 is rolling around the inside of a rectangle with side lengths 5 and 6. Given the total area swept out by the circle can be represented as $a + b\pi$ for positive integers a, b find $a + b$.



Solution: The shaded region the circle sweeps out is the entire rectangle, excluding a $(5-4)=1$ by $(6-4)=2$ rectangle in the middle, as well as 4 circular segments at each corner of the rectangle, where each circular segment is essentially a unit square but cutting off a quarter circle with center at one of the vertices of the unit square and radius 1). Hence, the desired area is $30-2-4(1-\frac{\pi}{4}) = 24+\pi$ for an answer of $a+b=\boxed{25}$.

Proposed by Bryan Li

10. **Problem:** Quadrilateral $ABCD$ has $\angle ABC = 90^\circ, \angle ADC = 120^\circ$, $AB = 5, BC = 18$, and $CD = 3$. Find AD.

    Solution: By the Pythagorean Theorem, we have $AC = \sqrt{5^2+18^2} = \sqrt{349}$. Letting $AD = x$, Law of Cosines on triangle ACD gives $\sqrt{x^2+9-2\cdot 3\cdot 3\cdot \cos 120^\circ} = \sqrt{349}$, or $x^2+9+3x = 349$. Subtracting both sides by 349 yields $x^2+3x-340=0$, and factoring gives $(x-17)(x+20)=20$, meaning $x=17$ or $x=-20$. Since side lengths cannot be negative, our answer is $\boxed{17}$.

    Solution 2: Extend let the altitude from C onto line AD be X. Since $\angle ADC = 120^\circ$, we know $\angle CDX = 60^\circ$. Hence, CDX is a $30-60-90$ triangle, meaing $DX = \frac{3}{2}$ and $CX = \frac{3\sqrt{3}}{2}$. Note that $AC^2 = 5^2+18^2$ by Pythagorean Theorem on $\triangle ABC$. Furthermore, $AC^2 = AX^2+CX^2 = (AD+\frac{3}{2})^2 + \left(\frac{3\sqrt{3}}{2}\right)^2$ by Pythagorean Theorem on $\triangle AXC$. Thus, $\left(AD+\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2 = 5^2+18^2$ which we can solve to find $AD = \boxed{17}$.

    Proposed by Raymond Gao

11. **Problem:** Raymond is eating huge burgers. He has been trained in the art of burger consumption, so he can eat one every minute. There are 100 burgers to start with. However, at the end of every 20 minutes, one of Raymond's friends comes over and starts making burgers. Raymond starts with 1 friend. If each of his friends makes 1 burger every 20 minutes, after how long in minutes will there be 0 burgers left for the first time?

    Solution:

At the end of 20 minutes, there are 81 left (his original friend has made 1), and 2 friends.

At the end of 40 minutes, there are 63 left (2 friends made 2), and 3 friends.

At the end of 60 minutes, there are 46 left (3 friends made 3), and 4 friends.

At the end of 80 minutes, there are 30 left (4 friends made 4), and 5 friends.

At the end of 100 minutes, there are 15 left (5 friends made 5), and 6 friends.

After 15 more minutes, Raymond has finally eaten all the burgers, giving the answer of $\boxed{115}$.

Proposed by Bryan Li

12. **Problem:** Find the number of pairs of positive integers (a, b) and $b \le a \le 2022$ such that $a \cdot \text{lcm}(a, b) = b \cdot \gcd(a, b)^2$.

    Solution: Let us only consider one prime at a time. Let $v_p(n)$ denote the largest power of p that divides n. Then, let $v_p(a) = a', v_p(b) = b'$.

    Case 1: $a' \le b'$. We know $v_p(a \cdot \text{lcm}(a, b)) = v_p(a) + \max(a', b') = a' + b'$. Additionally, $v_p(b \cdot \gcd(a, b)^2) = b' + 2\min(a', b') = b' + 2a'$. Thus, $a' + b' = b' + 2a' \Rightarrow a' = 0$.

    Case 2: $a' > b'$. Then, $v_p(a \cdot \text{lcm}(a, b)) = v_p(a) + \max(a', b') = a' + a'$ and $v_p(b \cdot \gcd(a, b)^2) = b' + 2\min(a', b') = b' + 2b' = 3b'$. Thus, $2a' = 3b'$.

    Therefore, for each prime p, either $v_p(a) = 0$ or $v_p(a) = \frac{3}{2} v_p b$ for $v_p(b) > 0$. Clearly, if $v_p(a) = \frac{3}{2} v_p(b)$ then $v_p(b)$ must be even, so $v_p(b) = 2k \Rightarrow v_p(a) = 3k$. Hence, a must be a cubic less than or equal to 2022. The cubes less than 2022 are $1^3, 2^3, \cdots, 12^3$.

    We can do casework on each cube to find the number of b that work for each case. In general, if $a = x^3$ then $b = k \cdot x^2$. We know $b \le a \Rightarrow k \le x$. Furthermore, $\gcd(k, x) = 1$. With this in mind, we find the total number of pairs (a, b) that work to be $\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(12) = \boxed{46}$.

    Proposed by Jerry Li

13. **Problem:** Triangle ABC has sides $AB = 6, BC = 10$, and $CA = 14$. If a point D is placed on the opposite side of AC from B such that $\triangle ADC$ is equilateral, find the length of BD.

    Solution: We will use Heron's formula on triangle ABC to find its area. The semiperimeter is 15, so the area is $\sqrt{15 \cdot 9 \cdot 5 \cdot 1} = 15\sqrt{3}$. Let the base of the perpendicular from A to BC be H. Then, we have $AH \cdot BC \div 2 = 15\sqrt{3}$, or $10 \div 2 \cdot AH = 15\sqrt{3}$, so $AH = 3\sqrt{3}$. Since triangle ABH is a right triangle, and $AB = 6$ and $AH = 3\sqrt{3}$, $\triangle ABH$ is a $30 - 60 - 90$ triangle with $\angle ABH = 60^\circ$. Furthermore, $\angle ABC = 180^\circ - \angle ABH = 120^\circ$. But, since $\angle ABC + \angle ADC = 120^\circ + 60^\circ = 180^\circ$ we know that quadrilateral $ABCD$ is cyclic. Letting $BD = x$ and applying Ptolemy's Theorem to the cyclic quadrilateral gives $14x = 14 \cdot 6 + 14 \cdot 10$. Solving for x gives $x = \boxed{16}$.

    Proposed by Raymond Gao

14. **Problem:** If the product of all real solutions to the equation $(x-1)(x-2)(x-4)(x-5)(x-7)(x-8) = -x^2 + 9x - 64$ can be written as $\frac{a - b\sqrt{c}}{d}$ for positive integers a, b, c, d where $\gcd(a, b, d) = 1$ and c is squarefree, compute $a + b + c + d$.

    Note: the problem originally said $(x-1)(x-2)(x-4)(x-5)(x-6)(x-7)(x-8)$, making the problem unsolvable in the way outlined below.

    Solution: Let $y = (x-2)(x-7) = x^2 - 9x + 14$. Substituting, the equation can we rewritten as $(y-6)y(y+6) = -y - 50$. Expanding and rearranging to form a cubic, we have $y^3 - 35y + 50 = 0$. We can easily see that $y = 5$ works, so we can factor to get

    $$(y-5)(y^2 + 5y - 10) = 0.$$

    Using the quadratic formula on the second factor, we have $y = 5, y = -\frac{5}{2} - \frac{\sqrt{65}}{2}$, or $y = -\frac{5}{2} + \frac{\sqrt{65}}{2}$.

Plugging these values of y into our original substitution equation, we have 3 cases:

Case 1: $x^2 - 9x + 14 = 5$, meaning $x^2 - 9x + 9 = 0$, and both roots are real, so by Vieta's, the product of the roots is 9.

Case 2: $x^2 - 9x + 14 = -\frac{5}{2} - \frac{\sqrt{65}}{2}$, but both solutions are not real, so we can disregard this case.

Case 3: $x^2 - 9x + 14 = -\frac{5}{2} + \frac{\sqrt{65}}{2}$, meaning $x^2 - 9x - \frac{\sqrt{65}}{2} + \frac{33}{2} = 0$. Both roots are real, so by Vieta's, the product of the roots is $\frac{33}{2} - \frac{\sqrt{65}}{2}$

Thus the product of all roots is $9 \cdot (\frac{33}{2} - \frac{\sqrt{65}}{2}) = \frac{297-9\sqrt{65}}{2}$, which yields $a + b + c + d = \boxed{373}$

Proposed by Bryan Li

15. **Problem:** Joe has a calculator with the keys $1, 2, 3, 4, 5, 6, 7, 8, 9, +, -$. However, Joe is blind. If he presses 4 keys at random, and the expected value of the result can be written as $\frac{x}{11^4}$, compute the last 3 digits of x when x divided by 1000. (If there are consecutive signs, they are interpreted as the sign obtained when multiplying the two signs values together, e.g $3, +, -, -, 2$ would return $3 + (-(-(2))) = 3 + 2 = 5$. Also, if a sign is pressed last, it is ignored.)

    Solution: Note that once Joe presses a sign (+ or -) the expected value is simply 0 after that. This is because he has an equal probability of pressing + or -, and these will cancel.

    Also notice that since we only care about expected value, we can replace all the digits 1-9 with their average, or 5. Now let us run casework on the number of digits pressed before a sign is pressed.

    Case 1: A sign is the first button pressed. This happens with probability $\frac{2}{11}$, and the expected value is 0 by the lemma.

    Case 2: The first sign is pressed on the 2nd keypress. This happens with probability $\frac{9}{11} \cdot \frac{2}{11} = \frac{18}{11^2}$, so the expected value is $5 \cdot \frac{18}{121}$ which would contribute $5 \cdot 18 \cdot 11^2 \pmod{1000}$ to x.

    Case 3: The first sign is pressed on the 3rd keypress. This happens with probability $(\frac{9}{11})^2 \cdot \frac{2}{11} = \frac{162}{11^3}$, so the expected value is $55 \cdot \frac{162}{11^3}$ which would contribute $55 \cdot 162 \cdot 11 \pmod{1000}$ to x.

    Case 4: The first sign is pressed on the 4th keypress. This happens with probability $(\frac{9}{11})^3 \cdot \frac{2}{11} = \frac{1458}{11^4}$, so the expected value is $555 \cdot \frac{1458}{11^4}$ which would contribute $555 \cdot 1458 \pmod{1000}$ to x.

    Case 5: No signs are pressed. This happens with probability $(\frac{9}{11})^4$, and the expected value is $5555 \cdot \frac{6561}{11^4}$ which would contribute $5555 \cdot 6561 \pmod{1000}$ to x.

    Via taking $\pmod{8}$ and $\pmod{125}$ on x, we find the last 3 digits of x to be $\boxed{445}$.

    Proposed by Bryan Li