Acton-Boxborough Math Competition Online Contest Solutions

Saturday, December 17 — Sunday, December 18, 2022

1. **Problem:** If $A = 0, B = 1, C = 2, \ldots, Z = 25$, then what is the sum of $A + B + M + C$?

   Solution: We know that A,B, and C are 0,1, and 2 respectively. To find the value of M, we can list out letters and their corresponding numbers: $D = 3, E = 4, F = 5, G = 6, H = 7, I = 8, J = 9, K = 10, L = 11$, and finally $M = 12$. So, the sum $A + B + M + C = 0 + 1 + 2 + 12 = \boxed{15}$.

   Proposed by Iris Shi

2. **Problem:** Eric is playing Tetris against Bryan. If Eric wins one-fifth of the games he plays and he plays 15 games, find the expected number of games Eric will win.

   Solution: The expected number of games that Eric will win is the probability that he wins one game multiplied by the number of games he plays, or $15 \cdot \frac{1}{5} = \boxed{3}$.

   Proposed by Raymond Gao

3. **Problem:** What is the sum of the measures of the exterior angles of a regular 2023-gon in degrees?

   Solution: The sum of the measures of the exterior angles of any polygon is always 360° regardless of the number of sides, so the sum of the measures of the exterior angles of a regular 2023-gon is $\boxed{360}$ degrees.

   Proposed by Daniel Cai

4. **Problem:** If N is a base 10 digit of $90N3$, what value of N makes this number divisible by 477?

   Solution: First, notice that 477 is a multiple of 9. This means that any multiple of it must be divisible by 9 as well, so we know $90N3$ must also be divisible by 9. We can now use the divisibility rule for 9 to get that $9 + N + 3 = 12 + N$ is a multiple of 9. So, N is clearly $\boxed{6}$. In order to check our answers, we can do long division to get that $9063 = 19 \cdot 477$ which further confirms that we are correct.

   Proposed by Ivy Shi

5. **Problem:** What is the rightmost non-zero digit of the decimal expansion of $\frac{1}{2^{2023}}$?

   Solution: As we are looking for the last non-zero digit, we can multiply by 10 any amount and it will not change as multiplication by 10 is simply adding a zero to the end. So, multiply this by 10^{2023} to get $\frac{10^{2023}}{2^{2023}} = 5^{2023}$. Now, we know that all powers of 5 end in 5 (for example, $5^2 = 25, 5^3 = 125, 5^4 = 625$) and so the last non-zero digit is $\boxed{5}$.

   Proposed by Iris Shi

6. **Problem:** If graphs of $y = \frac{5}{4}x + m$ and $y = \frac{3}{2}x + n$ intersect at $(16, 27)$, what is the value of $m + n$?

   Solution: We know that $y = \frac{5}{4}x + m$ and $y = \frac{3}{2}x + n$ intersect at $(16, 27)$. We can determine m and n by plugging $(16, 27)$ into both equations and then solving for the unknown values. Doing this gives us the equations $27 = \frac{5}{4}(16) + m$ and $27 = \frac{3}{2}(16) + n$. Solving for m and n gets us $m = 7$ and $n = 3$. Therefore, we get $m + n = 7 + 3 = \boxed{10}$.

   Proposed by Ivy Shi

7. **Problem:** Bryan is hitting the alphabet keys on his keyboard at random. If the probability he spells out ABMC at least once after hitting 6 keys is $\frac{a}{b^c}$, for positive integers a, b, c where b, c are both as small as possible, find $a + b + c$. Note that the letters ABMC must be adjacent for it to count: AEBMCC should not be considered as correctly spelling out ABMC.

   Solution: First of all, we must find the total number of ways Bryan can hit 6 alphabet keys (this will be our denominator before simplifying). There are 26 alphabet keys and so the total is 26^6. Now, we have three cases for spelling out ABMC: $xxABMC$, $xABMCx$,and $ABMCxx$ where the xs are any letter. Note that these three cases are clearly mutually exclusive as you cannot have more than 1 instance of $ABMC$ in six letters. So, the total number for each case is 26^2 and as there are three cases there will be $3 \cdot 26^2$ combinations. Therefore, the total probability is $\frac{3*26^2}{26^6} = \frac{3}{26^4}$. So, our answer is $3 + 26 + 4 = \boxed{33}$.

Proposed by Daniel Cai

8. **Problem:** It takes a Daniel twenty minutes to change a light bulb. It takes a Raymond thirty minutes to change a light bulb. It takes a Bryan forty-five minutes to change a light bulb. In the time that it takes two Daniels, three Raymonds, and one and a half Bryans to change 42 light bulbs, how many light bulbs could half a Raymond change? Assume half a person can work half as productively as a whole person.

   Solution: In terms of lightbulbs per hour, we know that Daniel's speed is 3 lph, Raymond's is 2 lph, and Bryan's is $\frac{4}{3}$ lph. So, the total speed of two Daniels, three Raymonds, and one and a half Bryans is $2 * 3 + 3 * 2 + \frac{3}{2} * \frac{4}{3} = 14$. The number of hours it will take them to change all 42 lightbulbs is $\frac{42}{14} = 3$ hours. Now, we know that half of a Raymond works at 2/2 = 1 lph. This means he will be able to change exactly $\frac{3}{1} = \boxed{3}$ lightbulbs in the same time.

   Proposed by Daniel Cai

9. **Problem:** How many integers between 1 and 1000 contain exactly two 1's when written in base 2?

   Solution: If the problem was for numbers from 1 to 1023, this would be really easy as it would just be all 10 digit base 2 numbers with leading digits allowed. This would mean the answer would be $\binom{10}{2} = 45$. However, we do not have the numbers from 1001 to 1023! But, this actually does not matter as these are all greater than $1110000000 = 896$ and hence their first three digits will all be 1s! So, none of the numbers from 1001 to 1023 have exactly two 1s when written in base 2. So, our answer is $\boxed{45}$.

   Proposed by Ivy Shi

10. **Problem:** Find the value of $5a + 4b + 3c + 2d + e$ given a, b, c, d, e are real numbers satisfying the following equations:

$$a^2 = 2e + 23$$
$$b^2 = 10a - 34$$
$$c^2 = 8b - 23$$
$$d^2 = 6c - 14$$
$$e^2 = 4d - 7.$$

   Solution: Add all five equations up. Now, we get that $a^2 + b^2 + c^2 + d^2 + e^2 - 10a - 8b - 6c - 4d - 2e + 55 = 0$. We can now complete the square to get that $(a-5)^2 + (b-4)^2 + (c-3)^2 + (d-2)^2 + (e-1)^2 = 0$. By the trivial inequality, all perfect squares are greater than or equal to zero with equality when the number inside the square is zero. So, $a - 5 = b - 4 = c - 3 = d - 2 = e - 1 = 0 \Rightarrow a = 5, b = 4, c = 3, d = 2, e = 1$. So, the answer is $5 * 5 + 4 * 4 + 3 * 3 + 2 * 2 + 1 * 1 = \boxed{55}$

   Proposed by Daniel Cai

11. **Problem:** Joe has lost his 2 sets of keys. However, he knows that he placed his keys in one of his 12 mailboxes, each labeled with a different positive integer from 1 to 12. Joe plans on opening the

mailbox labeled 1 to see if any of his keys are there. However, a strong gust of wind blows by, opening mailboxes 11 and 12, revealing that they are empty. If Joe decides to open one of the mailboxes labeled 2,3,4,5,6,7,8,9, or 10, the probability that he finds at least one of his sets of keys can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the sum $a+b$. Note that a single mailbox can contain 0, 1, or 2 sets of keys, and the mailboxes his sets of keys were placed in are determined independently at random.

Solution: Say Joe decides to open the mailbox labelled 2. It doesn't matter which maibox Joe opens - the probability remains the same for each mailbox labelled from 2 to 10. Say Joe has 2 sets of keys k_1, k_2. We know these keys cannot be in mailbox 11 or 12, so k_1, k_2 can both be placed in 10 different mailboxes, for 100 different total possibilities. We need at least one of k_1, k_2 to be placed in mailbox 2 for Joe to find at last one of his set of keys. This occurs in 19 ways. Thus, the desired probability is $\dfrac{19}{100}$, so $a+b=\boxed{119}$.

Proposed by Daniel Cai

12. **Problem:** As we all know, the top scientists have recently proved that the Earth is a flat disc. Bob is standing on Earth. If he takes the shortest path to the edge, he will fall off after walking 1 meter. If he instead turns 90 degrees away from the shortest path and walks towards the edge, he will fall off after 3 meters. Compute the radius of the Earth.

    Solution: Let's label points. We have a chord $\overline{DB}$, with B being Bob's position right now, and D being this closest possible location to fall off. Thus, $BD=1$. We also have chord $\overline{ABC}$, with $\overline{AC}$ perpendicular to $\overline{DB}$,, A and C on the circle's edge, and $AB=3$. We must realize here that $\overline{DB}$ is a portion of the diameter that is the Earth's circle. To understand this, visualize a line l tangent to the circle and intersecting it at D. Since it is tangent, $\overline{BD}$ is perpendicular to this line, meaning that $\overline{AB}$ is parallel to this line. All three of these cases only work if B is at the midpoint of $\overline{AC}$, which means that DB is part of the diameter. After this, let's assign the radius as x, and the center of the circle as O. Construct $\triangle ABO$. $BO=x-1, AB=3$, and $AO=x$. Using the Pythagorean theorem, $(x-1)^2+3^2=x^2$. From there, it is easy to find that $x=5$, so the radius of the Earth is $\boxed{5}$ meters.

    Proposed by Bryan Li

13. **Problem:** There are 999 numbers that are repeating decimals of the form 0.abcabcabc.... The sum of all of the numbers of this form that do not have a 1 or 2 in their decimal representation can be expressed as $\dfrac{a}{b}$ for relatively prime positive integers a, b. Find $a+b$.

    Solution: Note that $0.\overline{abc}$ is equivalent to $\frac{100a+10b+c}{999}$. We are only allowed to use the 8 digits $0,3,4,5,6,7,8,9$. The average of these digits is $\frac{42}{8}=\frac{21}{4}$. With these 8 digits, the total number of numbers we can form is $8\cdot 8\cdot 8=512$. Then, it suffices to find the average value of a term and multiply it by 512. Note that the average occurs when $a=b=c=\frac{21}{4}$. Then, we have an answer of $\frac{(100+10+1)\frac{21}{4}}{999}\cdot 512=\frac{111}{999}\cdot\frac{21}{4}\cdot 512=\frac{1}{9}\cdot\frac{21}{4}\cdot 512=\dfrac{896}{3}$. Hence, $a+b=\boxed{899}$.

    Proposed by Raymond Gao

14. **Problem:** An ant is crawling along the edges of a sugar cube. Every second, it travels along an edge to another adjacent vertex randomly, interested in the sugar it notices. Unfortunately, the cube is about to be added to some scalding coffee! In 10 seconds, it must return to its initial vertex, so it can get off and escape. If the probability the ant will avoid a tragic doom can be expressed as $\dfrac{a}{3^{10}}$, where a is a positive integer, find a.

    Solution: We can categorize the vertices into four groups - the initial vertex, the neighbors of the initial vertex, the neighbors of the opposite of the initial vertex, and the opposite of the initial vertex.

Let these groups be group 1,2,3,4 respectively. To avoid on unnecessary fractional math later, we can look at the number of ways to reach each group, and divide by 3^{10} at the end. If we let $a_{(g,t)}$ be the number of ways to reach group g at time t, we get that

$$\begin{aligned} a_{(1,t)} &= a_{(2,t-1)} \\ a_{(2,t)} &= 3a_{(1,t-1)} + 2a_{(3,t-1)} \\ a_{(3,t)} &= 3a_{(4,t-1)} + 2a_{(2,t-1)} \\ a_{(4,t)} &= a_{(3,t-1)}. \end{aligned}$$

Using these equations, we create the table below:

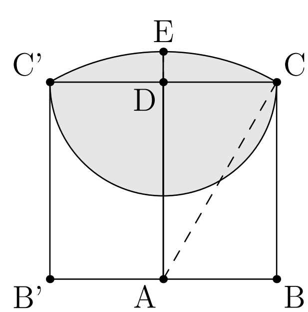
Time	Group 1	Group 2	Group 3	Group 4
0	1	0	0	0
1	0	3	0	0
2	3	0	6	0
3	0	21	0	6
4	21	0	60	0
5	0	183	0	60
6	183	0	546	0
7	0	1641	0	546
8	1641	0	4920	0
9	0	14763	0	4920
10	14763	0	44286	0

Because we want to end at a vertex in Group 1, the probability is $\frac{14763}{3^{10}}$ for an answer of $\boxed{14763}$.

Proposed by Eric Chen

15. **Problem:** Raymond's new My Little Pony: Friendship is Magic Collector's book arrived in the mail! The book's pages measure $4\sqrt{3}$ inches by 12 inches, and are bound on the longer side. If Raymond keeps one corner in the same plane as the book, what is the total area one of the corners can travel without ripping the page? If the desired area in square inches is $a\pi + b\sqrt{c}$ where a, b, and c are integers and c is squarefree, find $a + b + c$.

Solution:



We can visualize the problem by imagining that AD must remain stationary. If we move point C to some other point X, we need $DX \leq CD$ and $AX \leq AC$.

The desired area is a semicircle with radius $4\sqrt{3}$ and a segment of a circle with radius $8\sqrt{3}$ and bound by 60°. The area of the semicircle is 24π. The area of the segment of the circle is $(8\sqrt{3})^2\pi \cdot \frac{1}{6} - \frac{(8\sqrt{3})^2\sqrt{3}}{4} = 32\pi - 48\sqrt{3}$. Thus, the total area is $56\pi - 48\sqrt{3}$, so $a + b + c = \boxed{11}$.

Proposed by Eric Chen and Matthew Qian