Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 23 — Sunday, October 24, 2021

1. **Problem:** How many perfect squares are in the set: $\{1, 2, 4, 9, 10, 16, 17, 25, 36, 49\}$?

   Solution: The numbers 1, 4, 9, 16, 25, 36 and 49 are the perfect squares in this set. Thus, the number of perfect squares is $\boxed{7}$.

   Proposed by Alice Hui

2. **Problem:** If $a \odot b = a^b - ab - 5$, what is the value of $2 \odot 11$?

   Solution: $2^{11} = 2048$ and $2 \cdot 11 = 22$. $2048 - 22 - 5 = 2048 - 27 = \boxed{2021}$

   Proposed by Ryon Das

3. **Problem:** Joe can catch 20 fish in 5 hours. Jill can catch 35 fish in 7 hours. If they work together, and the amount of days it takes them to catch 900 fish is represented by $\frac{m}{n}$, where m and n are relatively prime integers, what is $m + n$?

   Assume that they work at a constant rate without taking breaks and that there are an infinite number of fish to catch.

   Solution: Joe catches 4 fish per hour and Jill catches 5 fish per hour. Together, they catch 9 fish per hour. It will take them $\dfrac{900}{9} = 100$ hours to catch 900 fish and $\dfrac{100}{24} = \dfrac{25}{6}$ days to catch 900 fish, so $m + n = \boxed{31}$.

   Proposed by Ryon Das

4. **Problem:** What is the units digit of 187^{10}?

   Solution: To find the units digit after a series of multiplication, we do not need to consider the tens place, and only the units place. We see that the units digit repeats itself, namely in the pattern $7 - 9 - 3 - 1$. Using this pattern, we find the units digit of 187^{10} is the same as the units digit as 187^2 which is $\boxed{9}$.

   Proposed by Alice Hui

5. **Problem:** What is the largest number of regions we can create by drawing 4 lines in a plane?

   Solution: When the nth line is added to the $(n - 1)$ lines, in order to form the maximum number of regions, the new line being added must intersect all of the already existing $n - 1$ lines; when this is done, n new regions are formed. One line can divide a plane into two regions, two non-parallel lines can divide a plane into 4 regions and three non-parallel lines can divide into 7 regions, and so on. Thus, the largest number of regions that we can create by drawing 4 lines in a plane is $\boxed{11}$.

   Proposed by Lakshika Kamalaganesh

6. **Problem:** A regular hexagon is inscribed in a circle. If the area of the circle is 2025π, given that the area of the hexagon can be expressed as $\dfrac{a\sqrt{b}}{c}$ for positive integers a, b, c where $\gcd(a, c) = 1$ and b is not divisible by the square of any number other than 1, find $a + b + c$.

   Solution: We first solve for the radius of the circle. Let r be the radius of the circle. We know that the area of a circle is πr^2. Setting this quantity equal to 2025π yields $r = 45$. Since the regular hexagon is composed of 6 equilateral triangles, all of which have side lengths equal to its circumcircle, we know that the regular hexagon has side length 45. The area of a regular hexagon is $\dfrac{3s^2\sqrt{3}}{2}$ where s is the side length of the hexagon. Since $s = 45$ we find the area of the hexagon to be $\dfrac{6075\sqrt{3}}{2}$. Hence, $a + b + c = \boxed{6080}$.

Proposed by Avanish Gowrishankar

7. **Problem:** Find the number of trailing zeroes in the product $3! \cdot 5! \cdot 719!$.

   Solution: We see that $3! \cdot 5! \cdot 719!$ evaluates to $6 \cdot 120 \cdot 719!$ which gives $720 \cdot 719!$ or $720!$. The amount of trailing zeroes in a factorial is just summing the floors of the number by the powers of five. $720/5 = 144$, which has a floor of 144 as well. $720/25 = 28.8$, with a floor of 28. $720/125$ has a floor of 5, and the floor of the quotient of 720 by 625 is 1. $144 + 28 + 5 + 1 = \boxed{178}$

   Proposed by Ryon Das

8. **Problem:** How many ordered triples (x, y, z) of odd positive integers satisfy $x + y + z = 37$?

   Solution: Let $x' = \frac{x+1}{2}$, $y' = \frac{y+1}{2}$ and $z' = \frac{z+1}{2}$. Thus, x', y' and z' are positive integers that satisfy $x' + y' + z' = 20$. Now we have established a one-to-one correspondence between the triples such that the number of ordered triples that satisfy the new sum must equal the number of ordered triples of the original statement. Choosing x', y' and z' is equivalent to choosing 2 dividers in the 19 slots provided. Thus, the answer is $\binom{19}{2} = \boxed{171}$.

   Proposed by Lakshika Kamalaganesh

9. **Problem:** Let N be a number with 2021 digits that has a remainder of 1 when divided by 9. $S(N)$ is the sum of the digits of N. What is the value of $S(S(S(S(N))))$?

   Solution: We first show that $S(S(S(S(N)))) \leq 9$. Note that $S(N) \leq 9 \cdot 2021 = 18189$ since each digit of the 2021 digits can at most be 9. Since $S(N)$ is less than or equal to 5 digits we know that $S(S(N)) \leq 5 \cdot 9 = 45$ since each digit of the 5 digits is at most 9. Since $S(S(N))$ is at most 2 digits we know that $S(S(S(N))) \leq 2 \cdot 9 = 18$. Note that the sum of the digits of any number less than or equal to 18 is less than or equal to 9 so $S(S(S(S(N)))) \leq 9$.

   Now note that $S(x) \equiv x \pmod 9$ for any number x. Since N leaves a remainder of 1 when divided by 9 we know that $N \equiv S(N) \equiv S(S(N)) \equiv S(S(S(N))) \equiv S(S(S(S(N)))) \equiv 1 \pmod 9$. Furthermore, we know that $S(S(S(S(N)))) \leq 9$. The only positive integer that is $1 \pmod 9$ and is less than or equal 9 is 1. Thus $S(S(S(S(N)))) = \boxed{1}$.

   Proposed by Alice Hui

10. **Problem:** Ayana rolls a standard die 10 times. If the probability that the sum of the 10 die is divisible by 6 is $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m + n$?

    Solution: Note that after any 9 rolls of the die, there is only one number from $1 - 6$ of the tenth dice that yields a final sum that is divisible by 6. Hence, the probability that the sum of the results of the 10 dice is divisible by 6 is $\frac{1}{6}$ so $m + n = \boxed{7}$.

    Proposed by Alice Hui

11. **Problem:** In triangle ABC, AB=13, BC=14, and CA=15. The inscribed circle touches the side BC at point D. The line AI intersects side BC at point K given that I is the incenter of triangle ABC. What is the area of the triangle KID?

    Solution:

    We know $\triangle KID$ is a right triangle since KD is tangent to the incircle at D. To find the area of the right triangle, we find the lengths of ID and KD.

    Clearly, ID is just the radius of the incircle. To find the inradius of the $13 - 14 - 15$ triangle we use the formula $s \cdot r = [\triangle ABC]$ where s is semiperimeter of $\triangle ABC$ and $[\triangle ABC]$ denotes the area of

$\triangle ABC$. The semiperimeter can easily be found to be 21. The area can be found to be 84 by splitting the $13-14-15$ triangle into a $5-12-13$ and a $9-12-15$ triangle. Hence, the radius of incircle is $\dfrac{84}{21} = 4$.

To find the length of KD we use $KD = BK - DK$. Let $x = BK$ and $14 - x = KC$. By the angle bisector theorem we know that $\dfrac{13}{x} = \dfrac{15}{14-x}$. Solving for x we find $BK = \dfrac{13}{2}$. To find BD we use the relatively well known formula $BD = \dfrac{AB + BC - AC}{2} = 6$. Hence, we have $DK = \dfrac{13}{2} - 6 = \dfrac{1}{2}$.

We know that $ID = 4$ and $KD = \dfrac{1}{2}$. Using the formula of a right triangle we find $[\triangle ABC] = \dfrac{ID \cdot KD}{2} = \boxed{1}$.

Proposed by Alice Hui

12. **Problem:** Given the cubic equation $2x^3 + 8x^2 - 42x - 188$, with roots a, b, c, evaluate $|a^2b + a^2c + ab^2 + b^2c + c^2a + bc^2|$.

    Solution: Of a cubic formula with $ax^3 + bx^2 + cx + d$, Vieta's formulas tell us that with the roots of the function being x_1, x_2, x_3, then $x_1 + x_2 + x_3 = -\frac{b}{a}$, $x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$, and $x_1x_2x_3 = -\frac{d}{a}$. We can apply these formulas to the problem. If we add $3abc$ to $a^2b + a^2c + ab^2 + b^2c + c^2a + bc^2$, we see that $a^2b + abc + a^2c + ab^2 + b^2c + abc + abc + c^2a + bc^2$ just factors to $(ab + bc + ca)(a + b + c)$. Vieta's formulas tell us for this certain cubic, $a + b + c = -4$, $ab + bc + ca = -21$, and $abc = 94$. Using our factored form, we get $(-4 \cdot -21) - 3(94)$, which evaluates to -198. Taking the absolute value yields $\boxed{198}$.

    Proposed by Ryon Das

13. **Problem:** In tetrahedron $ABCD$, AB=6, BC=8, CA=10, and DA, DB, DC=20. If the volume of $ABCD$ is $a\sqrt{b}$ where a, b are positive integers and in simplified radical form, what is $a + b$?

    Solution: Clearly, $\triangle ABC$ is a right triangle, and D is some point not on plane ABC. Let the altitude from D intersect plane ABC at X and have length h. Since $DA = DB = DC$ we know that triangles DAX, DBX, DCX all have congruent hypotenuses. Consider triangles DAX, DBX, DCX. All three triangles are right triangles, share a leg DX, and have congruent hypotenuses. Hence, $\triangle DAB \cong \triangle DBX \cong \triangle DCX$. Thus, we know that $AX = BX = CX$ implying that X is the circumcenter of $\triangle ABC$. Since ABC is a right triangle we know that $AX = BX = CX = \dfrac{AC}{2} = 5$. Applying Pythagorean theorem on any of the aforementioned right triangles, we obtain $5^2 + h^2 = 20^2 \Rightarrow h = 5\sqrt{15}$.

    The volume of a tetrahedron can be expressed as $\dfrac{Bh}{3}$ where B is the area of the triangle base and h is the height of the tetrahedron. The area of the triangle base is $\dfrac{6 \cdot 8}{2} = 24$ and we found $h = 5\sqrt{15}$. Hence, the volume of the tetrahedron is $\dfrac{24 \cdot 5\sqrt{15}}{3} = 40\sqrt{15} \Rightarrow a + b = \boxed{55}$.

    Proposed by Alice Hui

14. **Problem:** A 2021-digit number starts with the four digits 2021 and the rest of the digits are randomly chosen from the set 0,1,2,3,4,5,6. If the probability that the number is divisible by 14 is $\frac{m}{n}$ for relatively prime positive integers m, n, what is $m + n$?

    Solution: In order for the number to be divisible by 14 the number needs to be divisible by both 2 and 7. For the number to be divisible by 2, the last digit must be even. Since 4 of the 7 available digits are even, the probability that the last digit is even occurs with probability $\frac{4}{7}$.

Choose any random even digit for the last digit, and choose random digits for all the other unchosen digits except for the second to last digit. Let the second to last digit be x and let the sum of the rest of the number be y. We need $10x + y \equiv 0 \pmod 7 \Rightarrow 3x + y \equiv 0 \pmod 7 \Rightarrow x \equiv 2y \pmod 7$. Note that no matter what y is, we will always only have 1 number x that works in the range $0 - 6$ meaning that the probability the rest of the number is divisible by 7 is $\frac{1}{7}$. Hence, the probability the number is divisible by 14 is $\frac{4}{7} \cdot \frac{1}{7} = \frac{4}{49} \Rightarrow m + n = \boxed{53}$.

Proposed by Alice Hui

15. **Problem:** Let $ABCD$ be a cyclic quadrilateral with circumcenter O_1 and circumradius 20. Let the intersection of AC and BD be E. Let the circumcenter of $\triangle EDC$ be O_2. Given that the circumradius of $\triangle EDC$ is 13, $O_1O_2 = 11$, $AB = 16$, $BE = \frac{26\sqrt{2}}{3}$, find O_1E^2.

Solution: Note that $O_1O_2 \perp DC$ since both points are on the perpendicular bisector of DC. Let line O_1O_2 intersect DC at M, the midpoint of DC, and assign x to O_2M and y to DM. By applying Pythagorean theorem, we know that $x^2 + y^2 = 169$ and $(x + 11)^2 + y^2 = 400$. Subtracting the first equation from the second, we obtain $22x + 121 = 231 \Rightarrow 110 \Rightarrow x = 5$. Solving for y, we find $y = 12$. Since $y = DM$ is half of the of the length of DC we find $DC = 2 \cdot 12 = 24$.

Now we show that $\triangle DO_1O_2 \sim \triangle CBE$ (Credits to Kevin Zhao). First, note that $\angle DO_1O_2 = \angle DAC = \angle DBC = \angle EBC$ where the first equality comes from the fact that $\angle DO_1O_2 = \frac{1}{2}\angle DO_1C = \frac{1}{2}(2\angle DAC) = \angle DAC$. Furthermore, we know that $\angle DO_2O_1 = 180° - \angle DO_2m = 180° - \frac{1}{2}\angle DO_2C = 180° - \angle DEC = \angle BEC$. Thus, by AA similiarity, we find $\triangle DO_1O_2 \sim \triangle CBE$. Using this similarity, we find $\frac{O_1O_2}{BE} = \frac{DO_2}{EC} \Rightarrow \frac{11}{11\sqrt{2}} = \frac{13}{EC} \Rightarrow EC = 13\sqrt{2}$. Since $\triangle EO_2C$ is a $13 - 13 - 13\sqrt{2}$ triangle we find $\angle EO_2C = 90°$.

With the information that $\angle EO_2C = 90°$ we can deduce that $\angle O_1O_2E = 180° - 90° - \angle CO_2M = 90 - \angle CO_2M = \angle O_2CD$. Note that in $\triangle O_1O_2E$ we have $O_1O_2 = 11$ and $O_2 = 13$ and $\cos(\angle O_1O_2E) = \cos(\angle O_2CD) = \frac{12}{13}$. Hence, applying law of cosines to O_1O_2E or dropping an altitude from O_1 onto O_2E we have

$$O_1E^2 = O_1O_2^2 + O_2E^2 - 2 \cdot O_1 \cdot O_2 \cdot \cos(\angle O_1O_2E) = 11^2 + 13^2 - 2 \cdot 11 \cdot 13 \cdot \frac{12}{13} = \boxed{26}.$$

Proposed by Jerry Li