Acton-Boxborough Math Competition Online Contest Solutions

Saturday, November 20 — Sunday, November 21, 2021

1. **Problem:** Martin's car insurance costed \$6000 before he switched to Geico, when he saved 15% on car insurance. When Mayhem switched to Allstate, he, a safe driver, saved 40% on car insurance. If Mayhem and Martin are now paying the same amount for car insurance, how much was Mayhem paying before he switched to Allstate?

   Solution: $6000(100-15)/100 = 0.6M$, where M is Mayhem's insurance cost before switching. Solving this gives us $M = \boxed{8500}$

   Proposed by Avanish Gowrishankar

2. **Problem:** The 7-digit number N can be written as $\underline{A}\ \underline{2}\ \underline{0}\ \underline{B}\ \underline{2}\ \underline{1}\ \underline{5}$. How many values of N are divisible by 9?

   Solution: The sum of the digits must be divisible by 9. We see $A+2+0+B+2+1+5 = 10+A+B$ must be divisible by 9. Since A and B can each be 9 at most, $10+9+9 = 28$ is the maximum sum of the digits of N. The sum of the digits must be 18 or 27 ($11 < \text{sum} \leq 28$), so $A+B$ must be 8 or 17. A can be any integer from 1 to 8, and A can only be 9 or 8 to sum to 17. Hence, the total number values of N that are divisible by 9 is $8+2 = \boxed{10}$.

   Proposed by Alice Hui

3. **Problem:** The solutions to the equation $x^2 - 18x - 115 = 0$ can be represented as a and b. What is $a^2 + 2ab + b^2$?

   Solution: Factoring this gives us $(x-23)(x+5) = 0$, making the roots 23 and -5. Factoring $a^2 + 2ab + b^2$ gives us $(a+b)^2$. Plugging this in yields $(23-5)^2 = \boxed{324}$.

   Proposed by Ryon Das

4. **Problem:** The exterior angles of a regular polygon measure to 4 degrees. What is a third of the number of sides of this polygon?

   Solution: The sum of all exterior angles in a polygon is 360°. Since a regular polygon would divide this by n, if n is the number of sides in the polygon, $360/n$ is 4, so $n = 90$. Hence, one third of the number of sides is $90/3 = \boxed{30}$.

   Proposed by Ryon Das

5. **Problem:** Charlie Brown is having a thanksgiving party.

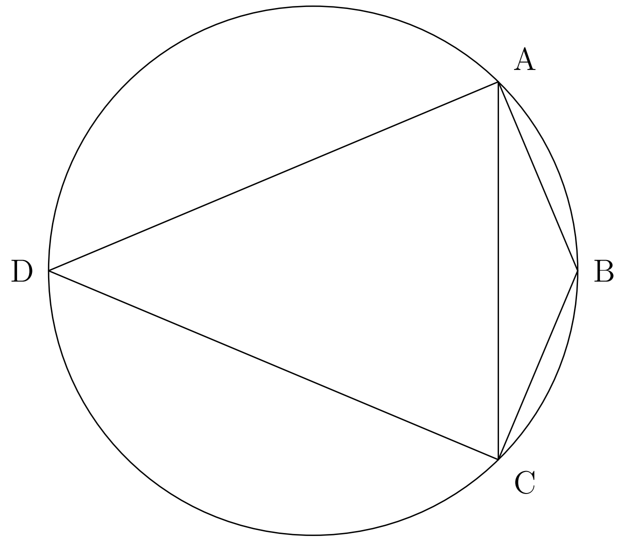
   - He wants one turkey, with three different sizes to choose from.
   - He wants to have two or three vegetable dishes, when he can pick from Mashed Potatoes, Sautéed Brussels Sprouts, Roasted Butternut Squash, Buttery Green Beans, and Sweet Yams;
   - He wants two desserts out of Pumpkin Pie, Apple Pie, Carrot Cake, and Cheesecake.

   How many different combinations of menus are there?

   Solution: There are three ways to choose the size of turkey. There are $\binom{5}{3}+\binom{5}{2} = 10+10 = 20$ ways to choose vegetables. There are $\binom{4}{2}$ = 6 ways to choose desert. Multiplying these together, we find $3 \cdot 20 \cdot 6 = \boxed{360}$

   Proposed by Ryon Das

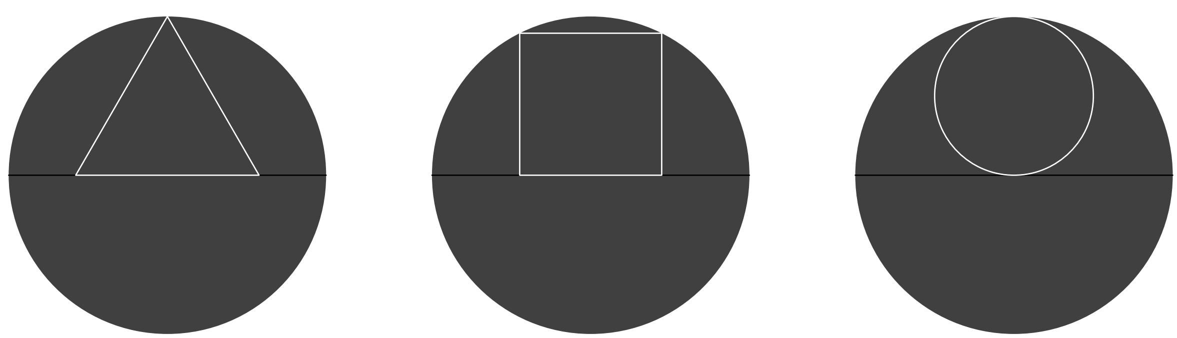
6. **Problem:** In the diagram below, $\overline{AD} \cong \overline{CD}$ and $\triangle DAB$ is a right triangle with $\angle DAB = 90°$. Given that the radius of the circle is 6 and $m\angle ADC = 30°$, if the length of minor arc $\widehat{AB}$ can be written as $a\pi$, what is a?



Solution: $\angle DCA$ is $(180^\circ - 30^\circ)/2 = 75^\circ$ since $\triangle ADC$ is isosceles. $\angle ACB$ is $90^\circ - 75^\circ = 15^\circ$. The Inscribed Angle Theorem tells us that arc $\widehat{AB}$ is 30 degrees. The circumference of the circle is $2 * 6\pi = 12\pi$. Since $\widehat{AB}$ subtends a 30° angle, we find the length to be $30/360 * 12\pi = 1\pi \Rightarrow a = \boxed{1}$.

Proposed by Avanish Gowrishankar

7. **Problem:** This Halloween, Owen and his two friends dressed up as guards from Squid Game. They needed to make three masks, which were black circles with a white equilateral triangle, circle, or square inscribed in their upper halves. Resourcefully, they used black paper circles with a radius of 5 inches and white tape to create these masks. Ignoring the width of the tape, how much tape did they use? If the length can be expressed $a\sqrt{b} + c\sqrt{d} + \frac{e}{f} \cdot \pi$ such that b and d are not divisible by the square of any prime, and e and f are relatively prime, find $a + b + c + d + e + f$.



Solution: Let's start with the triangle in the semicircle. If we drop an angle bisector from the vertex at the top, we create a 30-60-90 triangle, because of the fact that an equilateral triangle has all measures 60°. Recall that in a 30-60-90 triangle, if the side opposite the 30° angle has a measure of x, the side opposite the 90° angle has a measure of $2x$, and the side opposite the 60° angle has a measure of $x\sqrt{3}$. Applying this to the diagram, where we have the side opposite the 60° angle equalling to 5, then we have the side opposite the 30° have a measure of $\frac{5}{\sqrt{3}}$, and the side opposite the 90° have a measure of $\frac{10}{\sqrt{3}}$, which rationalizes to $\frac{10\sqrt{3}}{3}$. To find the perimeter, we just need to multiply by 3 to get $10\sqrt{3}$.

Now, let's work on the square in the semicircle. If we take the midpoint of the bottom segment and connect it to the top-right vertex, we have a right triangle, where we can apply the Pythagorean

Theorem to. The leg sides have half the bottom side and the whole right side, so can assign x to one leg and $2x$ to the other. Therefore, using Pythagorean, we get $x^2 + (2x)^2 = 5^2$, which simplifies to $5x^2 = 25$, or $x = \sqrt{5}$. We need to double that to get the side length, and then quadruple it to get the perimeter, which comes to $8\sqrt{5}$.

Our last case is very simple. The radius of the big circle, 5, is equal to the diameter of the small circle, which means the radius of the inscribed circle is 2.5. Using the perimeter of a Circle formula, the perimeter is 5π, or $\frac{5}{1}\pi$.

Hence, $a + b + c + d + e + f = 10 + 3 + 8 + 5 + 5 + 1 = \boxed{32}$.

Proposed by Eric Chen

8. **Problem:** Given $LCM(10^8, 8^{10}, n) = 20^{15}$, where n is a positive integer, find the total number of possible values of n.

   Solution: Let $n = 2^{p_1} \cdot 5^{p_2}$. Note that n cannot be divisible by any other prime, since the least common multiple is only divisible by 2 and 5. We also find $10^8 = 2^8 \cdot 5^8, 8^{10} = 2^{30}$, and $20^{15} = 2^{30} \cdot 5^{15}$. We see that p_1 can only take values from 0 to 31, and p_2 must equal 15, for a total of $\boxed{31}$ solutions.

   Proposed by Alice Hui

9. **Problem:** If one can represent the infinite progression $\frac{1}{11} + \frac{2}{13} + \frac{3}{121} + \frac{4}{169} + \frac{5}{1331} + \frac{6}{2197}...$ as $\frac{a}{b}$, where a and b are relatively prime positive integers, what is a?

   Solution:

   We split up the desired summation $\frac{1}{11} + \frac{2}{13} + \frac{3}{121} + \frac{4}{169} + \frac{5}{1331} + \frac{6}{2197}...$ into the following: $\left(\sum_{n=0} \frac{2n-1}{11^n}\right) + \left(\sum_{n=0} \frac{2n}{13^n}\right)$. We evaluate each part separately.

   Let $S_{11} = \sum_{n=1} \frac{2n-1}{11^n}$. We find $11S_{11} = 1 + \sum_{n=1} \frac{2n+1}{11^n}$. Subtracting the equation for S_{11} from the equation for $11S_{11}$ we find $10S_{11} = 1 + 2\sum_{n=1} \frac{1}{11^n}$. We can easily find $\sum_{n=1} \frac{1}{11^n} = \frac{1}{10}$ so $10S_{11} = 1 + \frac{1}{5} \Rightarrow S_{11} = \frac{3}{25}$.

   Let $S_{13} = \sum_{n=1} \frac{2n}{13^n}$. We find $13S_{13} = 2 + \sum_{n=1} \frac{2n+2}{13^n}$. Subtracting the equation for S_{13} from the equation for $13S_{13}$ we find $12S_{13} = 2 + 2\sum_{n=1} \frac{1}{13^n}$. We easily find $\sum_{n=1} \frac{1}{13^n} = \frac{1}{12}$ so $12S_{13} = 2 + \frac{1}{6} \Rightarrow S_{13} = \frac{13}{72}$.

   We are looking for $S_{11} + S_{13} = \frac{3}{25} + \frac{13}{72} = \frac{\boxed{541}}{1800}$.

   Proposed by Ryon Das

10. **Problem:** Consider a tiled 3x3 square without a center tile. How many ways are there to color the squares such that no two colored squares are adjacent (vertically or horizontally)? Consider rotations of an configuration to be the same, and consider the no-color configuration to be a coloring.

    Solution: For simplicity, denote the 4 corner squares as **corners** and the other 4 squares as **centers**. Proceed with casework on how many squares are colored. Either 4, 3, 2, 1, or 0 squares can be colored.

    If we color 4 squares, then every other square must be colored, giving 2 possibilities (either the corners or the centers)

If we color 3 squares, then we either color:

- 3 corners
- 2 corners and a center
- 1 corner and 2 centers
- 3 centers

There is only 1 unique configuration for 3 corners. For 2 corners and a center, if the two corners and opposite one another, then there is no possible center to use as well. So, the two corners must be adjacent to one another, leaving 1 possible center. For 1 corner and 2 centers, once we pick the corner, the 2 centers we choose are fixed, giving 1 possibility due to rotational symmetry. For 3 centers, there is again only 1 possibility. So, this case yields 4 ways.

For 2 squares, we can color:

- 2 corners
- 1 corner and 1 center
- 2 centers

For 2 corners, they can either be adjacent or opposite one another, giving two ways. For 1 corner and 1 center, there are 2 possibilities for the center once we choose a corner. We cannot get to one configuration to the other using rotation, so they are distinct. For 2 centers, they can also be adjacent or opposite, giving 2 more ways. so, this case yields 6.

For 1 square, we either color 1 corner or 1 center, both of which have 1 possibility each, giving 2.

For 0 squares, there is the empty configuration.

So, adding together all of the cases, we get:

$$2 + 4 + 6 + 2 + 1 = \boxed{15}.$$

Proposed by Jerry Li

11. **Problem:** Let ABC be a triangle with $AB = 4$ and $AC = 7$. Let AD be an angle bisector of triangle ABC. Point M is on AC such that AD intersects BM at point P, and $AP : PD = 3 : 1$. If the ratio $AM : MC$ can be expressed as $\dfrac{a}{b}$ such that a, b are relatively prime positive integers, find $a + b$.

    Solution: From the Angle Bisector Theorem, we know that $BD : DC = 4 : 7$. Because triangles APB and PBD share a common altitude, and because $AP : PD = 3 : 1$, we know that $A_{APB} : A_{PBD} = 3 : 1$. If we draw in segment PC, notice that triangles PBD and BPC share a common altitude, and therefore $A_{PBD} : A_{BPC} = 4 : 7$. From these two, we know that $A_{ABD} : A_{BPC} = 3 : \frac{11}{4}$. We know that $A_{APM} : A_{MPC} = A_{ABM} : A_{CBM} = AM : MC$, so therefore $A_{PBD} : A_{BPC} = AM : MC = \frac{3}{\frac{11}{4}} = \frac{12}{11}$, so the answer is $12 + 11 = \boxed{23}$.

Proposed by Eric Chen

12. **Problem:** For a positive integer n, define $f(n)$ as the number of positive integers less than or equal to n that are coprime with n. For example, $f(9) = 6$ because 9 does not have any common divisors with 1, 2, 4, 5, 7, or 8.

    Calculate:
$$\sum_{i=2}^{100} \left(29^{f(i)} \mod i\right).$$

    Solution: Notice that the function f is simply the Totient Function, So, by Euler's Totient Theorem, this expression is congruent to $1 \mod i$ whenever i and 29 are coprime. This holds for all i except $29, 58, 87$.

    Hence, we only have to consider these 3 cases.

    When $i = 29$, we have
$$29^{f(29)} \mod 29 \equiv 0.$$

    When $i = 58$, first off, realize that $f(58) = 28$. So, the expression equals:
$$29^{28} \mod 58.$$

    By The Chinese Remainder Theorem, we can consider mod 2 and 29.

    Since 29^{28} is odd, it is 1 mod 2. Trivially, it is also 0 mod 29. So, the expression is congruent to 29 mod 58.

    Lastly, we consider $i = 87$. $f(87) = 56$ so we have:
$$29^{56} \mod 87.$$

    Again, using CRT, we only have to consider mod 3 and 29. $29 \equiv -1 \mod 3$ so $29^{56} \equiv (-1)^{56} \equiv 1$ mod 3. Further, it is 0 mod 29. Thus, $29^{56} \equiv 58 \mod 87$.

    Since for all other i, the expression is 1, the desired sum is $1 \cdot 96 + 0 + 29 + 58 = \boxed{183}$.

    Proposed by Advay Goel

13. **Problem:** Let ABC be an equilateral triangle. Let P be a randomly selected point in the incircle of ABC. Find $a + b + c + d$ if the probability that $\angle BPC$ is acute can be expressed as $\frac{a\sqrt{b} - c\pi}{d\pi}$ for positive integers a, b, c, d where $\gcd(a, c, d) = 1$ and b is not divisible by the square of any prime.

    Solution:

    Let the points of tangency of the incircle and $\triangle ABC$ be D, E, O where D is on $\overline{AB}$ and E is on $\overline{AC}$. Note that to find the probability that $\angle BPC$ is acute we find the area in which $\angle BPC$ is acute, divided by the area of the incircle. WLOG that $\triangle ABC$ has side length 2. The area in which $\angle BPC$ is acute consists of the area outside of circle with diameter BC and inside the incircle. Call the circle with diameter BC circle ω, and called the desired area X.

    First note that D, E lie on ω. To prove this, simply note that $\angle BEC = \angle CDB = 90^\circ$ since $\triangle ABC$ is equilateral. Now let the area formed by $\overline{BD}$, arc $\widehat{DE}$, $\overline{EC}$, $\overline{BC}$ be Y. Let the area formed by $\overline{AD}$, $\overline{AE}$, arc $\widehat{DE}$ be Z. Note that X can be found by subtracting Y and Z from $[ABC]$ where brackets denote area. Let K be the area formed by arc $\widehat{BD}$, $\overline{BD}$.

We can find Y by subtracting $2 \cdot K$ from the area of the semicircle with diameter BC. The area of the semicircle with diameter BC is $\frac{\pi}{2}$. Note that the area of K is simply the area of sector BOD minus $[BOD]$. Hence, $K = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$, and

$$Y = \frac{\pi}{2} - 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{6} + \frac{\sqrt{3}}{2}.$$

We can find Z by finding $[ADFE]$ minus the area of sector DIE where I is the incenter of $\triangle ABC$. Note that $[ABC] = \sqrt{3}$ and $[ADFE]$ is one third of $[ABC]$ which is $\frac{\sqrt{3}}{3}$. The area of sector DIE is one third of the area of the incircle. Note that the inradius is $\frac{1}{\sqrt{3}}$ so the area of sector DIE is $\frac{1}{3} \cdot \pi r^2 = \frac{\pi}{9}$. Hence, we have

$$Z = \frac{\sqrt{3}}{3} - \frac{\pi}{9}.$$

Note that

$$X = [ABC] - Y - Z = \sqrt{3} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{3} - \frac{\pi}{9}\right) = \frac{\sqrt{3}}{6} - \frac{\pi}{18}.$$

From our work we know that the area of the incircle is $\pi r^2 = \frac{1}{3}\pi$. Hence, the desired probability is

$$\frac{\frac{\sqrt{3}}{6} - \frac{\pi}{18}}{\frac{1}{3}\pi} = \frac{\frac{\sqrt{3}}{2} - \frac{\pi}{6}}{\pi} = \frac{3\sqrt{3} - \pi}{6\pi} \Rightarrow a + b + c + d = \boxed{13}.$$

Proposed by Jerry Li

14. **Problem:** When the following expression is simplified by expanding then combining like terms, how many terms are in the resulting expression?

$$(a+b+c+d)^{100} + (a+b-c-d)^{100}$$

Solution: Let $x = a + b$ and $y = c + d$. Then, the expression becomes

$$(x+y)^{100} + (x-y)^{100}.$$

Every other term cancels out in this expansion, leaving:

$$\sum_{i=0}^{50} 2 \cdot \binom{100}{2i} x^{2i} y^{100-2i}.$$

We only care about the number of terms, so the binomial coefficient is irrelevant and will not be included for the rest of the solution. From the definition of x and y, we can rewrite $x^{2i}y^{100-2i}$ as

$$(a+b)^{2i}(c+d)^{100-2i} = \sum_{j=0}^{2i} \sum_{k=0}^{100-2i} a^j b^{2i-j} c^k d^{100-2i-k}.$$

Notice that every single term produced will be different as we never never the same quartet of exponents on a, b, c, d. So, for each i, we produce $(2i+1)(101-2i)$ terms. Because i ranges from 0 to 50, the total number of terms in the expansion becomes:

$$\sum_{i=0}^{50}(2i+1)(101-2i) = 1\cdot 101 + 3\cdot 99 + 5\cdot 97 + \cdots + 99\cdot 3 + 101\cdot 1.$$

From the difference of squares, we obtain:

$$(51^2-50^2)+(51^2-48^2)+\cdots+(51^2-2^2)+(51^2-0^2)+(51^2-2^2)+\cdots+(51^2-50^2).$$

Simplification yields:

$$51\cdot 51^2 - 2\sum_{i=1}^{25}(2i)^2 = 51^3 - 8\sum_{i=1}^{25} i^2 = 51^3 - 8\cdot\frac{25\cdot 26\cdot 51}{6} = \boxed{88451}.$$

Proposed by Alice Hui

15. **Problem:** Jerry has a rectangular box with integral side lengths. If 3 units are added to each side of the box, the volume of the box is tripled. What is the largest possible volume of this box?

Solution: Let x, y, z be the integer side lengths of the rectangular box. From the given, we know that $(x+3)(y+3)(z+3) = 3xyz$ which we write as

$$\left(\frac{x+3}{x}\right)\left(\frac{y+3}{y}\right)\left(\frac{z+3}{z}\right) = 3.$$

We show that at least one of x, y, z must be less than or equal to 6. Assume that $x, y, z \geq 7$. The maximum value of $\left(\frac{x+3}{x}\right)\left(\frac{y+3}{y}\right)\left(\frac{z+3}{z}\right) = 3$ comes from maximizing each term individually. Clearly, if we have $x \geq 7$ then when $x = 7$ we have the maximum value for $\frac{x+3}{x}$. Using similar logic for the other terms, we find that the maximum value of the expression $\left(\frac{x+3}{x}\right)\left(\frac{y+3}{y}\right)\left(\frac{z+3}{z}\right) = 3$ is $\frac{10}{7}\cdot\frac{10}{7}\cdot\frac{10}{7} = \frac{1000}{343}$. However, note that $3 > \frac{1000}{343}$, so we have

$$3 > \frac{1000}{343} \geq \left(\frac{x+3}{x}\right)\left(\frac{y+3}{y}\right)\left(\frac{z+3}{z}\right) = 3$$

which is a contradiction to the fact that $\left(\frac{x+3}{x}\right)\left(\frac{y+3}{y}\right)\left(\frac{z+3}{z}\right) = 3.$

Let x be less than or equal to 7. We now do casework on x.

Case 1: $x = 6$.

Plugging $x = 6$ into our equation, we find

$$\begin{aligned}
9\cdot(y+3)(z+3) &= 18yz\\
(y+3)(z+3) &= 2yz\\
yz + 3z + 3y + 9 &= 2yz\\
9 &= yz - 3y - 3z\\
18 &= (y-3)(z-3)
\end{aligned}$$

where we used Simon's Favorite Factoring Trick for the last line. Note that we wish to maximize $xyz = 6yz$, which is obviously maximized when yz is maximized. Solving for yz, we see $yz = 9 + 3(y + z)$ so we should obviously maximize $y + z$. Note that $y + z$ is maximized when y, z are furthest apart, so we have $y - 3 = 18, z - 3 = 1 \Rightarrow\ y = 21, z = 4$. The maximum volume of the box for this case is then $6 \cdot 21 \cdot 4 = 504$.

Case 2: $x = 5$.

Plugging $x = 5$ into our equation, we find

$$\begin{aligned} 8(y+3)(z+3) &= 15yz \\ 8yz + 24y + 24z + 72 &= 15yz \\ 72 &= 7yz - 24y - 24z \\ 504 &= 49yz - 168y - 168z \\ 1080 &= (7y - 24)(7y - 24) \end{aligned}$$

where we once again used Simon's Favorite Factoring Trick for the last step. Once again, to maximize yz we should aim to maximize $y + z$ which occurs when y, z are furthest apart. Note that when we set $7y - 24 = 1, 7y - 24 = 1080$ we have no integral solutions. Similarly, the pairs $(2, 540), (3, 360)$ also do not work. However, we see that setting $7y - 24 = 4, 7y - 24 = 270$ we will obtain an integral pair $(y, z) = (4, 42)$. Thus, the maximized volume is $5 \cdot 4 \cdot 42 = 840$ for this case.

Case 3: $x = 4$.

Plugging $x = 4$ into our equation, we find

$$\begin{aligned} 7(y+3)(z+3) &= 12yz \\ 7yz + 21y + 21z + 63 &= 12yz \\ 63 &= 5yz - 21y - 21z \\ 315 &= 25yz - 105y - 105z \\ 315 &= (5y - 21)(5z - 21) \\ 756 &= (5y - 21)(5z - 21). \end{aligned}$$

We see that yz is maximized when y, z are furthest apart. We once again test cases. Note that letting $5y - 21 = 1, 5z - 21 = 756$ gives no integral pairs. Similarly, letting $5y - 21 = 2, 5z - 21 = 378$ or $5y - 21 = 3, 5z - 21 = 252$ both do not give integral pairs. However, letting $5y - 21 = 4, 5y - 21 = 189$ yields the integral pair $(y, z) = (5, 42)$. Thus, the maximum volume for this case would be $5 \cdot 42 \cdot 4 = 840$.

Case 4: $x = 3$.

Plugging $x = 3$ into our equation, we find

$$\begin{aligned} 6(y+3)(z+3) &= 9yz \\ 2(y+3)(z+3) &= 3yz \\ 2yz + 6y + 6z + 18 &= 3yz \\ 18 &= yz - 6y - 6z \\ 54 &= (y - 6)(z - 6). \end{aligned}$$

Once again yz is maximized when y, z are furthest apart. This occurs when $y - 6 = 1, z - 6 = 54 \Rightarrow y = 7, z = 60$. Hence, the maximum volume is $3 \cdot 7 \cdot 60 = 1260$.

Case 5: $x = 2$.

Plugging $x = 2$ into our equation, we find

$$\begin{aligned} 5(y+3)(z+3) &= 6yz \\ 5yz + 15y + 15z + 45 &= 6yz \\ 45 &= yz - 15y - 15z \\ 270 &= (y-15)(z-15). \end{aligned}$$

Since yz is maximized when y, z are furthest apart, we let $y - 15 = 270, z - 15 = 1 \Rightarrow \ (y, z) = 285, 16$. Hence, the maximum volume for this case is $2 \cdot 16 \cdot 285 = 9120$.

Case 6: $x = 1$.

There are no solutions to this case. To see this, note that our equation simplifies to $4(y+3)(z+3) = 3yz$. Clearly, since y, z are positive integers, we have $\frac{y+3}{y}, \frac{z+3}{z} > 1 \Rightarrow \ 4 \cdot \frac{y+3}{y} \cdot \frac{z+3}{z} > 4$. This contradicts the fact that $4 \cdot \frac{y+3}{y} \cdot \frac{z+3}{z} = 3$ so there are no solutions to this case.

From our cases, we see when $x = 2$ we obtain the largest volume, namely $\boxed{9120}$.

Proposed by Alice Hui