Acton-Boxborough Math Competition Online Contest Solutions

Saturday, December 18 — Sunday, December 19, 2021

1. **Problem:** In rectangle ABMC, AB= 5 and BM= 8. If point X is the midpoint of side AC, what is the area of triangle XCM?

   Solution: Let's start with the fact that parallelograms have opposite sides congruent and equal. This holds true for rectangles, as all rectangles are parallelograms. Therefore, with rectangle ABMC, we have $AB = MC = 5$ and $BM = AC = 8$. Midpoints imply that congruent segments are formed, and that a smaller segment created by a midpoint is half of the big segment. Therefore, $XC = \frac{1}{2}AC$, so $XC = 4$. All rectangles contain four right angles. Since $\angle C$ is the meeting of two of the rectangles sides, it is right, and all triangles containing right angles are right triangles. In right triangles the legs are the two sides which are not opposite to the right angle. To calculate the area of a right triangle, we can multiply the legs and divide by 2. It is clear that in triangle XCM, XC and MC are the legs. To find the area, we compute $(XC \cdot MC)/2 = (4 \cdot 5)/2 = 20/2 = \boxed{10}$.

   Proposed by Alice Hui

2. **Problem:** Find the sum of all possible values of $a + b + c + d$ such that (a, b, c, d) are quadruplets of (not necessarily distinct) prime numbers satisfying $a \cdot b \cdot c \cdot d = 4792$.

   Solution: Note that $4792 = 2^3 \cdot 599$. We see that 3 of the 4 values of a, b, c, d must be 2. Consequently, the final prime number must be 599. Hence, the only possible value of $a + b + c + d$ is $2 + 2 + 2 + 599 = \boxed{605}$.

   Proposed by Advay Goel

3. **Problem:** How many integers from 1 to 2022 inclusive are divisible by 6 or 24, but not by both?

   Solution: $\frac{2022}{6} = 337$. That means that there are 337 integers divisible by 6. Because 24 is a multiple of 6, every multiple of 24 is also a multiple of 6. In addition, since 24 is four times 6, one in every four multiples of 6 is also divisible by 4. To find, we lower 337 to the nearest multiple of 4, which is 336. We multiply this by $\frac{3}{4}$ to get 252. We discounted one multiple, so we add it back to get $\boxed{253}$.

   Proposed by Alice Hui

4. **Problem:** Jerry begins his English homework at 07:39 a.m. At 07:44 a.m., he has finished 2.5% of his homework. Subsequently, for every five minutes that pass, he completes three times as much homework as he did in the previous five minute interval. If Jerry finishes his homework at $AB : CD$ a.m., what is $A + B + C + D$? For example, if he finishes at 03:14 a.m., $A + B + C + D = 0 + 3 + 1 + 4$.

   Solution: As we see in the problem, Jerry has finished 2.5% of his homework at $7 : 44$ a.m. To see the speed Jerry works at for the next five minutes, we multiply his percentage per five minutes by three, to see that he finishes 7.5% in the next five minutes. Adding these percentages, we see he has finished a combined 10% of his English homework. $7.5 \cdot 3 = 22.5$, which is the percent he finishes in the five minutes ending at $7 : 54$. Finally, we multiply that percentage by three to get the percentage done in the next five minutes, 67.5%. Summing up the percentages per five minutes, $2.5 + 7.5 + 22.5 + 67.5 = 100$, showing that the full 100% of Jerry's English homework has been finished after the fourth five minute period. The fourth five minute period ends at $07 : 59$ a.m. $0 + 7 + 5 + 9 = \boxed{21}$.

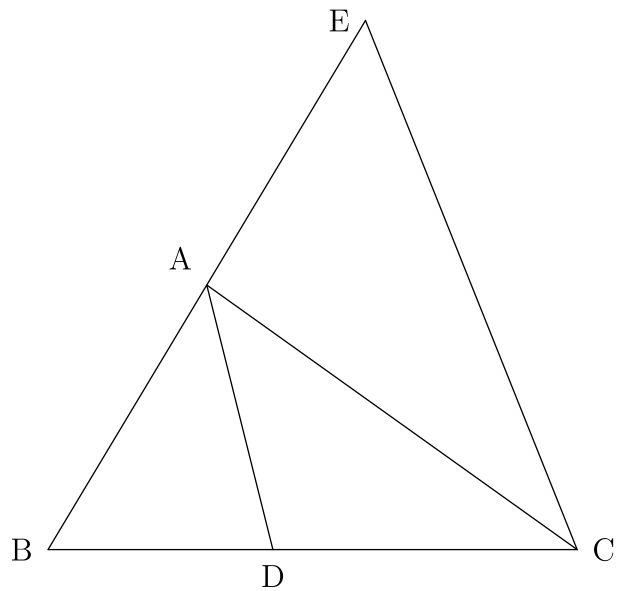
   Proposed by Avanish Gowrishankar

5. **Problem:** Advay the frog jumps 10 times on Mondays, Wednesdays and Fridays. He jumps 7 times on Tuesdays and Saturdays. He jumps 5 times on Thursdays and Sundays. How many times in total did Advay jump in November if November 17th falls on a Thursday? (There are 30 days in November).

Solution: We know that if November 17th is a Thursday, November 3rd is a Thursday, and November 1st is a Tuesday. By summing the number of jumps on each day, we find that in a week, Advay jumps $10+7+10+5+10+7+5=54$ times. There are four full weeks in November, so we can multiply to find that Advay jumps $54 \cdot 4 = 216$ times through the first four weeks. All that's left to calculate is the jumps on the 29th and 30th. Because the 1st is a Tuesday, the 29th is also a Tuesday and the 30th is a Wednesday. Summing these up, we see that Advay jumped $216+7+10=\boxed{233}$ times in November.

Proposed by Lakshika Kamalaganesh

6. **Problem:** In the following diagram, $\angle BAD \cong \angle DAC$, $\overline{CD} = 2\overline{BD}$, and $\angle AEC$ and $\angle ACE$ are complementary. Given that $\overline{BA} = 210$ and $\overline{EC} = 525$, find $\overline{AE}$.



Solution: Since $\angle AEC$ and $\angle ACE$ are complementary, we find $\angle EAC = 90^\circ$. By the Angle Bisector Theorem, we see $\dfrac{BA}{AC} = \dfrac{BD}{CD} \Rightarrow \dfrac{210}{AC} = \dfrac{1}{2} \Rightarrow AC = 420$. To find AE we find Pythagorean theorem triples. We find $EC = 21 \cdot 25$, $AC = 21 \cdot 20$, so $AE = 21 \cdot 15 = \boxed{315}$.

Proposed by Avanish Gowrishankar

7. **Problem:** How many trailing zeros are there when 2021! is expressed in base 2021?

Solution: Note that the number of trailing zeros in 2021! in base 2021 is the number of factors of 2021 in 2021! We see that there are $\left\lfloor \dfrac{2021}{47} \right\rfloor = 43$ factors of 47 in 2021! and $\left\lfloor \dfrac{2021}{43} \right\rfloor + \left\lfloor \dfrac{2021}{43^2} \right\rfloor = 48$ factors of 43 in 2021! Hence, in total there are $\min(43, 48) = 43$ factors of 2021 in 2021! for $\boxed{43}$ trailing zeros.

Proposed by Alice Hui

8. **Problem:** When two circular rings of diameter 12 on the Olympic Games Logo intersect, they meet at two points, creating a 60° arc on each circle. If four such intersections exist on the logo, the area of the regions of the logo that exist in at least two circles is $a\pi - b\sqrt{c}$ where a, b, c are integers and $\sqrt{c}$ is fully simplified find $a+b+c$.

Solution: Each intersection creates two areas that can be determined by subtracting the area of an isosceles triangle with legs of 6 from a 60° segment of a radius 6 circle. The area of the segment is

$\frac{60}{360}6^2\pi = 6\pi$. Because the vertex angle of the isosceles triangle is 60°, the base angles of the triangle must be 60°, making this an equilateral triangle. This makes the area of the triangle $9\sqrt{3}$. Each intersection creates two sectors of $6\pi - 9\sqrt{3}$ area. Since four such intersections exist, this area is $4 \cdot 2(6\pi - 9\sqrt{3}) = 48\pi - 72\sqrt{3}$. Hence, we find $a + b + c = 48 + 72 + 3 = \boxed{123}$.

Proposed by Avanish Gowrishankar

9. **Problem:** If $x^2 + ax - 3$ is a factor of $x^4 - x^3 + bx^2 - 5x - 3$, then what is the absolute value of $a + b$?

   Solution: Let's start by setting up values for the quotient of the division we currently have. Let's name it $rx^2 + sx + t$, where $(rx^2 + sx + t) \cdot (x^2 + ax - 3) = x^4 - x^3 + bx^2 - 5x - 3$. For simplicities sake, let's call $x^2 + ax - 3$ (I), $rx^2 + sx + t$ (II), and $x^4 - x^3 + bx^2 - 5x - 3$ (III). There are two things we know about (II).

   - The leading coefficients of (I) and (III) are 1, so the leading coefficient of (II) must be 1 as well
   - The constant of a product polynomial is the product of the constants in the factors. Therefore, with (I) having a constant of -3, (II) must have a constant of 1 such that (III) has a constant of -3 as well.

   We have two very helpful pieces of information. Now, let's substitute them into (II), getting $x^2 + sx + 1$. Restating our multiplication, we have $(x^2 + sx + 1) \cdot (x^2 + ax - 3) = x^4 - x^3 + bx^2 - 5x - 3$. Let's just multiply them out, and we get $x^2 \cdot x^2 + x^2 \cdot ax + x^2 \cdot sx + x^2 \cdot -3 + x^2 \cdot 1 + ax \cdot sx + ax \cdot 1 + sx \cdot -3 + 1 \cdot -3 = x^4 - x^3 + bx^2 - 5x - 3$. By expanding and then grouping these, we see that we get $(1)x^4 + (a + s)x^3 + (-3 + 1 + as)x^2 + (a - 3s)x - 3 = \text{(III)}$.

   Simplifying a bit more, we see we can create a system.

   - $a + s = -1$
   - $as - 2 = b$
   - $a - 3s = -5$

   By taking the top and bottom parts of the system, it's easily reached that $a = -2$ and $s = 1$. From there, by substituting, we find that $b = -4$. Now, we see that the full polynomial equation is $(x^2 + x + 1) \cdot (x^2 - 2x - 3) = x^4 - x^3 - 4x^2 - 5x - 3$. A quick check shows that this is valid and that all the initial requirements have been made. Therefore, we see that $a + b = -2 + -4 = -6$, with an absolute value of $\boxed{6}$.

   Proposed by Alice Hui

10. **Problem:** Let (x, y, z) be the point on the graph of $x^4 + 2x^2y^2 + y^4 - 2x^2 - 2y^2 + z^2 + 1 = 0$ such that $x + y + z$ is maximized. Find $a + b$ if $xy + xz + yz$ can be expressed as $\dfrac{a}{b}$ where a, b are relatively prime positive integers.

    Solution: Note that the expression in the numerator factors into $(x^2 + y^2 - 1)^2 + z^2 = 0$. The left hand side of the equation is only 0 when $x^2 + y^2 - 1 = 0, z^2 = 0 \Rightarrow x^2 + y^2 = 1, z = 0$. Note that $x + y + z$ is maximized when $x + y + 0 = x + y$ is maximized. To maximize $x + y$, first note it is optimal to have x, y be positive. Applying $AM - GM$ on x^2, y^2 we have $\dfrac{x^2 + y^2}{2} \geq xy \Rightarrow \dfrac{1}{2} \geq xy$. In addition, note that $(x + y)^2 = x^2 + y^2 + 2xy \leq 1 + 1 = 2 \Rightarrow x + y \leq \sqrt{2}$. Equality occurs when $x = y = \dfrac{\sqrt{2}}{2}$. Thus,

    $$xy + yz + xz = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot 0 + \frac{\sqrt{2}}{2} \cdot 0 = \frac{1}{2} \Rightarrow a + b = \boxed{3}.$$

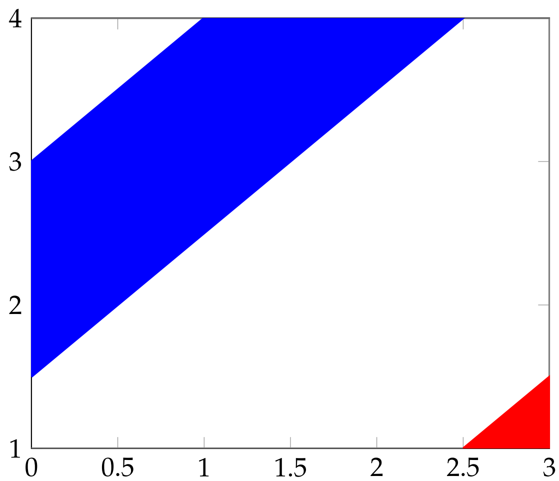
Proposed by Matthew Qian

11. **Problem:** Andy starts driving from Pittsburgh to Columbus and back at a random time from 12pm to 3pm. Brendan starts driving from Pittsburgh to Columbus and back at a random time from 1pm to 4pm. Both Andy and Brendan take 3 hours to get from one city to the other and back, and they travel at constant speeds. The probability that they pass each other closer to Pittsburgh than Columbus is m/n, for relatively prime positive integers m and n. What is $m+n$?

    Solution: The only way this can happen is that one of them must depart from Pittsburgh after the other departs from Columbus. That is, one of them must start at least 1.5 hours after the other. Furthermore, the times in which they depart cannot differ in more than 3 hours, since otherwise they would never meet. We can visualize this with a graph, shown below. The blue triangle represents the times where Andy leaves at least 1.5 hours before Brendan, and not exceeding 3 hours. The red triangle represents the time where Brendan leaves at least 1.5 hours before Andy. The starting times are chosen at random, so our probability is the fraction of the 3 by 3 square that is shaded. We get

$$\frac{\text{Area of blue triangle} + \text{Area of red triangle}}{\text{Area of square}} = \frac{\frac{25}{8} - \frac{4}{8} + \frac{1}{8}}{9} = \frac{11}{36}.$$

    So our answer is $11 + 36 = \boxed{47}$.



Proposed by Matthew Qian

12. **Problem:** Consider trapezoid $ABCD$ with AB parallel to CD and $AB < CD$. Let $AD \cap BC = O$, $BO = 5$, and $BC = 11$. Drop perpendicular AH and BI onto CD. Given that $AH : AD = \frac{2}{3}$ and $BI : BC = \frac{5}{6}$, calculate $a+b+c+d-e$ if $AB + CD$ can be expressed as $\dfrac{a\sqrt{b}+c\sqrt{d}}{e}$ where a,b,c,d,e are integers with $\gcd(a,c,e) = 1$ and $\sqrt{b}$, $\sqrt{d}$ are fully simplified.

    Solution: From $\dfrac{BI}{BC} = \dfrac{5}{6}, BC = 11$ we find $BI = \dfrac{55}{6}$. We the know that $AH = BI = \dfrac{55}{6}$. Using $\dfrac{AH}{AD} = \dfrac{2}{3} \Rightarrow AD = \dfrac{55}{4}$. We can then use Pythagorean Theorem on $\triangle ADH$ and $\triangle BCI$ to find $DH = \dfrac{55\sqrt{5}}{12}$ and $CI = \dfrac{11\sqrt{11}}{6}$. Let $AB = x$. Since $\triangle OAB \sim \triangle ODC$ we see that $\dfrac{AB}{DC} = \dfrac{OB}{OC} = \dfrac{5}{16}$. We see that $DC = x + \dfrac{55\sqrt{5}}{12} + \dfrac{11\sqrt{11}}{6} \Rightarrow x = \dfrac{25\sqrt{5}}{12} + \dfrac{5\sqrt{11}}{6}$. Note that

$$AB + CD = 2x + \frac{55\sqrt{5}}{12} + \frac{11\sqrt{11}}{6} = \frac{105\sqrt{5}}{12} + \frac{21\sqrt{11}}{6} = \frac{35\sqrt{5}}{4} + \frac{7\sqrt{11}}{2} = \frac{35\sqrt{5}+14\sqrt{11}}{4}$$

Thus, $a+b+c+d-e = \boxed{61}$.

Proposed by Advay Goel

13. **Problem:** The polynomials $p(x)$ and $q(x)$ are of the same degree and have the same set of integer coefficients but the order of the coefficients is different. What is the smallest possible positive difference between p(2021) and q(2021)?

Solution: Let $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{i=0}^{n} b_i x^i$, where the b_i are a permutation of the a_i. Then

$$p(2021) - q(2021) = \sum_{i=0}^{n}(a_i - b_i)2021^i \equiv \sum_{i=0}^{n}(a_i - b_i) \pmod{2020}.$$

Note that $\sum_{i=0}^{n} a_i - b_i = \sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i = 0$ since the b_i are a permutation of the a_i, so $p(2021) - q(2021) \equiv 0 \pmod{2020}$. Then the smallest possible positive difference is $\boxed{2020}$, which is achieved for $p(x) = 2x+1$ and $q(x) = x+2$.

Proposed by Jerry Li

14. **Problem:** Let $ABCD$ be a square with side length 12, and P be a point inside $ABCD$. Let line AP intersect DC at E. Let line DE intersect the circumcircle of ADP at $F \neq D$. Given that line EB is tangent to the circumcircle of ABP at B, and $FD = 8$, find $m+n$ if AP can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n.

Solution: Let $EC = x$. We then know that $ED = 12 - x$. Note that by power of a point, we have $EB^2 = EP \cdot EA = ED \cdot EF$. From Pythagorean Theorem, we can find $EB^2 = x^2 + 144$, and we find $ED \cdot EF = (12-x) \cdot (20-x)$. Hence, we have $x^2 + 144 = (12-x)(20-x) \Rightarrow x = 3$. Note that $ED = 12 - x = 9$, so using Pythagorean Theorem on $\triangle EAD$ yields $EA = 15$. Note that $EB^2 = EP \cdot EA \Rightarrow 12^2 + 3^2 = EP \cdot 15 \Rightarrow EP = \frac{153}{15} = \frac{51}{5}$. Hence, $AP = AE - EP = 15 - \frac{51}{5} = \frac{24}{5} \Rightarrow m+n = \boxed{29}$.

Proposed by Jerry Li

15. **Problem:** A three digit number m is chosen such that its hundreds digit is the sum of the tens and units digits. What is the smallest natural number n such that n cannot divide m?

Solution: Let's see which n don't work. We see that for $1 \leq a \leq 9$, the number $110a$ satisfies the condition that the hundreds digit is the sum of the tens and units digits. Furthermore, m is is divisible by $a, 2a, 5a, 11a$, so looping from $a = 1$ to $a = 0$ this eliminates, for $n \leq 23$: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 22. This leaves 13, 17, 19, 21, 23. We can check each of them using modular arithmetic: Let $m = 100x + 10y + z$. Since $x = y + z$, $m = 110y + 101z$. Now we can take this mod m. For $m = 13$, we get $m \equiv 6y + 10z \pmod{m}$. We see that $y = 1, z = 2$ makes 13 divide m. So 312 is a counterexample to 13. In the same way, we can eliminate 17, 19, and 21: 17 divides 918, 19 divides 532, and 21 divides 945. However, when we try this on $m = 23$, we get $18y + 9z \equiv 0 \pmod{23}$. This is the same as $2y + z \equiv 0 \pmod{23}$. Since $y < 10$ and $y + z = x < 10$, we have $2y + z < 20$, and so it is impossible for m to be divisible by 23. Thus $\boxed{23}$ is our answer.

Proposed by Matthew Qian