Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 17 — Sunday, October 18, 2020

1. **Problem:** Catherine's teacher thinks of a number and asks her to subtract 5 and then multiply the result by 6. Catherine accidentally switches the numbers by subtracting 6 and multiplying by 5 to get 30. If Catherine had not swapped the numbers, what would the correct answer be?

   Solution: Let x be the original number. We know that Catherine subtracts 6 and multiplies by 5 and get 30 as an answer, so we must have $5(x-6) = 30$. Solving this gives $x = 12$. Next, we perform the correct operations to compute $6(x-5)$ which gives $\boxed{42}$ as an answer.

   Proposed by Anusha Senapati

2. **Problem:** At Acton Boxborough Regional High School, desks are arranged in a rectangular grid-like configuration. In order to maintain proper social distancing, desks are required to be at least 6 feet away from all other desks. Assuming that the size of the desks is negligible, what is the maximum number of desks that can fit in a 25 feet by 25 feet classroom?

   Solution: The optimal configuration is a 5 by 5 square grid of desks, which spans $4 \cdot 6 = 24$ feet in both dimensions since the desks are assumed to have negligible size. Thus the answer is $\boxed{25}$.

   Proposed by Jerry Tan

3. **Problem:** Joshua hates writing essays for homework, but his teacher Mr. Meesh assigns two essays every 3 weeks. However, Mr. Meesh favors Joshua, so he allows Joshua to skip one essay out of every 4 that are assigned. How many essays does Joshua have to write in a 24-week school year?

   Solution: If Mr. Meesh assigns two essays every three weeks, then he will assign $2 \cdot \dfrac{24 \text{ weeks}}{3 \text{ weeks}} = 16$ essays over the school year. Joshua can skip $\dfrac{1}{4}$ of them, so he must write $\dfrac{3}{4} \cdot 16 = \boxed{12}$ essays during the school year.

   Proposed by Annie Wang

4. **Problem:** Libra likes to read, but she is easily distracted. If a page number is even, she reads the page twice. If a page number is an odd multiple of three, she skips it. Otherwise, she reads the page exactly once. If Libra's book is 405 pages long, how many pages in total does she read if she starts on page 1? (Reading the same page twice counts as two pages.)

   Solution: We begin with the number of pages she reads if she reads each page exactly once and then add or subtract the pages she reads twice or skips. We find she would have read 405 pages with no other conditions. She reads 202 pages corresponding to even page numbers twice. She skips pages $3, 9, \ldots, 405$, which totals $\dfrac{405-3}{6} + 1 = 68$ pages. Thus, in total, she reads $405 + 202 - 68 = \boxed{539}$ pages.

   Proposed by Devin Brown

5. **Problem:** Let the GDP of an integer be its ***G***reatest ***D***ivisor that is ***P***rime. For example, the GDP of 14 is 7. Find the largest integer less than 100 that has a GDP of 3.

   Solution: Begin by noticing that if the GDP of a number is 3, it must have a prime factorization of the form $2^a \cdot 3^b$ where $a \geq 0$ and $b \geq 1$. We can start by considering the case $b = 1$, and we find that the maximum number of this form less than 100 is $96 = 2^5 \cdot 3$. The only multiple of 3 greater than 96 and less than 100 is 99, and we see that 99 does not work since it is a multiple of 11. Thus, $\boxed{96}$ is the answer.

   Proposed by Jerry Tan

6. **Problem:** As has been proven by countless scientific papers, the Earth is a flat circle. Bob stands at a point on the Earth such that if he walks in a straight line, the maximum possible distance he can

travel before he falls off is 7 miles, and the minimum possible distance he can travel before he falls off is 3 miles. Then the Earth's area in square miles is $k\pi$ for some integer k. Compute k.

Solution: Let Earth's center be the point O. Let Bob be standing on point B. Extend line OB past B to intersect the circle again at C. We claim that the path BC is the shortest path for Bob to fall off Earth. Assume for the sake of contradiction that BD is the shortest path for some point $D \neq C$. By the triangle inequality on $\triangle OBD$, we have $OB + BD > DO$ or $BD > DO - OB = CO - OB = BC$. Thus BC is the shortest possible path.

Extend BO past O to intersect the circle again at E. A similar argument shows that BE is the longest possible path. Thus, the diameter $EC = 7 + 3 = 10$, so Earth's area is $\boxed{25}\pi$.

Proposed by Ethan Han

7. **Problem:** Edward has 2 magical eggs. Every minute, each magical egg that Edward has will double itself. But there's a catch. At the end of every minute, Edward's brother Eliot will come outside and smash one egg on his forehead, causing Edward to lose that egg permanently. For example, starting with 2 eggs, after one minute there will be 3 eggs, then 5, 9, and so on. After 1 hour, the number of eggs can be expressed as $a^b + c$ for positive integers a, b, c where $a > 1$, and a and c are as small as possible. Find $a + b + c$.

   Solution: We claim that, after t minutes, Edward will have $2^t + 1$ eggs. We prove this by induction. The base case is $t = 0$. Edward starts with $2 = 2^0 + 1$ eggs, so the base case satisfies our claim. After k minutes, if Edward has $2^k + 1$ eggs, each of them will double after another minute, producing $2^{k+1} + 2$ eggs. Eliot will smash one of the eggs on his forehead, so Edward will have $2^{k+1} + 1$ eggs after $k + 1$ minutes, so the inductive step holds as well. Therefore, after 60 minutes, Edward will have $2^{60} + 1$ eggs, so the desired result is $2 + 60 + 1 = \boxed{63}$.

   Proposed by Ethan Kuang

8. **Problem:** Define a sequence of real numbers $a_1, a_2, a_3, \ldots, a_{2019}, a_{2020}$ with the property that $a_n = \dfrac{a_{n-1} + a_n + a_{n+1}}{3}$ for all $n = 2, 3, 4, 5, \cdots 2018, 2019$. Given that $a_1 = 1$ and $a_{1000} = 1999$, find a_{2020}.

   Solution: The recurrence can be simplified in the following manner:

   $$a_n = \frac{a_{n-1} + a_n + a_{n+1}}{3} \Rightarrow 3a_n = a_{n-1} + a_n + a_{n+1} \Rightarrow 2a_n = a_{n-1} + a_{n+1} \Rightarrow a_{n+1} - a_n = a_n - a_{n-1}.$$

   Therefore, the difference between consecutive terms of the sequence is constant, so the sequence must be an arithmetic sequence.

   Let $a_2 = x$. From this recurrence we can find that $a_3 = 2x - 1$. Therefore, the common difference is $x - 1$, and $a_n = (n-1)x - (n-2)$ for all $n \geq 2$. We prove this last statement by induction. The base case is $n = 2$ and $a_2 = x = (2-1)x - (2-2)$, so the base case holds. If $a_k = (k-1)x - (k-2)$, then $a_{k+1} = a_k + x - 1 = kx - (k-1)$, so the inductive step holds as well.

   From this we see that $a_{1000} = 999x - 998 = 1999 \Rightarrow x = 3$. As a result, we have $a_{2020} = 2019x - 2018 = 2019 \cdot 3 - 2018 = \boxed{4039}$.

   Proposed by Jerry Li

9. **Problem:** In $\triangle ABC$ with $AB = 10$ and $AC = 12$, points D and E lie on sides $\overline{AB}$ and $\overline{AC}$, respectively, such that $AD = 4$ and $AE = 5$. If the area of quadrilateral BCED is 40, find the area of $\triangle ADE$.

   Solution: First draw the segment BE. Let $[BEC] = 7x$. Since $\triangle ABE, \triangle BEC$ share the same altitude length, and $\dfrac{AE}{EC} = \dfrac{5}{7}$ we know that $[ABE] = 5x$. Furthermore, we know that $\triangle BED$ and $\triangle AED$ share an altitude, so the ratio of the areas is the ratio of the sides. Since $\dfrac{BD}{AD} = \dfrac{3}{2}$ and $[BDE] + [ADE] = 5x$

we know that $[BDE] = 3x$ and $[ADE] = 2x$. Finally, note that $[BECD] = 10x = 40$ so $x = 4$. We want to find $[ADE] = 2x = \boxed{8}$.

Proposed by Jerry Tan

10. **Problem:** A positive integer is called *powerful* if every prime in its prime factorization is raised to a power greater than or equal to 2. How many positive integers less than 100 are powerful?

    Solution: We do casework on the lowest prime in the prime factorization.

    Case 1 : 2 is the lowest prime.

    For the integer to be powerful, we must have the factor of 2 be $4, 8, 16, 32, 64$ in the prime factorization for 5 numbers that are powerful. We can also multiply these numbers by powers of 3 greater than 2, as long as we multiply by at least two factors of 3. That gives us 2 more powerful numbers, namely $4 \cdot 9$ and $8 \cdot 9$ for 7 powerful numbers in total for this case. We see that $16 \cdot 9$ or $4 \cdot 27$ are both greater than 100, so there are no more powerful numbers less than 100 with just powers of 2 and 3. We cannot add in a factor of 5 since then the smallest possible number, $4 \cdot 25 = 100$, is not less than 100.

    Case 2 : 3 is the lowest prime.

    We must have the factor of 3 be $9, 27, 81$ for 3 numbers that are powerful. We cannot multiply any of these numbers by 25 since that would exceed 100.

    Case 3 : 5 is the lowest prime.

    There is only one number that works, namely 25.

    $Case 4$: 7 is the lowest prime.

    There is only one number that works, namely 49.

    All other possibilities exceed 100. Therefore the total number of powerful numbers is $7 + 3 + 1 + 1 = \boxed{12}$.

    Proposed by Devin Brown

11. **Problem:** Let integers $A, B < 10,000$ be the populations of Acton and Boxborough, respectively. When A is divided by B, the remainder is 1. When B is divided by A, the remainder is 2020. If the sum of the digits of A is 17, find the total combined population of Acton and Boxborough.

    Solution: We have two cases, either $B > A$ or $A > B$.

    Case 1 : $B > A$.

    Since $B > A$ we know that the remainder when A is divided by B is just A, so $A = 1$. However, this is a contradiction since, from the second condition, A must be greater than 2020 to leave a remainder of 2020. Therefore, there are no solutions in this case.

    Case 2 : $A > B$.

    If $A > B$ then the remainder when B is divided by A is just B, so $B = 2020$. Since $A \equiv 1 \pmod{B}$ we know that $A \equiv 1 \pmod{2020}$ which means $A = 2020k + 1$ for some positive integer k. Since $A < 10000$. the only possible values of A are $2021, 4041, 6061$,and 8081. Only 8081 satisfies the condition that the sum of the digits of A is 17. Hence, $A = 8081$ and $B = 2020$ for a total of $A + B = \boxed{10101}$.

    Proposed by Jerry Tan

12. **Problem:** Let $a_1, a_2, \cdots, a_n$ be an increasing arithmetic sequence of positive integers. Given $a_n - a_1 = 20$ and $a_n^2 - a_{n-1}^2 = 63$, find the sum of the terms in the arithmetic sequence.

    Solution: Let $a_1 = a$ and the common difference of the sequence be k. We know that $a_n = a + k(n-1)$. From $a_n - a_1 = 20$ we know that $k(n-1) = 20$ and from $a_n^2 - a_{n-1}^2 = 63$ we know that $(a + k(n-1))^2 - (a + k(n-2))^2 = (2a + k(2n-3))(k) = 63$. Since this sequence is an increasing sequence of positive integers we know that $a, k$and n are all integers. Therefore we know that $n - 1$

is an integer and $2a + k(2n - 3)$ is also an integer. This means that k is a divisor of both 20 and 63. Since $\gcd(20, 63) = 1$ we know that $k = 1$. From here, we find $n = 21$ and $a = 12$. Hence $a_n = 32$ and, applying the sum of an arithmetic sequence formula, we find that the desired answer is $\frac{(12+32)(21)}{2} = 22 \cdot 21 = \boxed{462}$.

Proposed by Jerry Li

13. **Problem:** Bob rolls a cubical, an octahedral and a dodecahedral die $(6, 8$ and 12 sides respectively) numbered with the integers from 1 to 6, 1 to 8 and 1 to 12 respectively. If the probability that the sum of the numbers on the cubical and octahedral dice equals the number on the dodecahedral die can be written as $\frac{m}{n}$, where m, n are relatively prime positive integers, compute $n - m$.

    Solution: There are 48 combinations of a roll on the 6-sided die and the 8-sided die. If the roll is anything except $(6, 8)$, $(6, 7)$, or $(5, 8)$, the sum of the two die will be at most 12 and at least 2. Thus, in $48 - 3 = 45$ cases, there is exactly one roll on the 12-sided die that matches the sum, and otherwise no roll on the 12-sided die works. Thus, there are 45 total successful combinations out of $6 \cdot 8 \cdot 12$ total, so the probability is $\frac{45}{6 \cdot 8 \cdot 12} = \frac{5}{64}$ and the answer is $64 - 5 = \boxed{59}$.

    Proposed by Ethan Han

14. **Problem:** Let $\triangle ABC$ be inscribed in a circle with center O with $AB = 13, BC = 14, AC = 15$. Let the foot of the perpendicular from A to BC be D and let AO intersect BC at E. Given the length of DE can be expressed as $\frac{m}{n}$ where m, n are relatively prime positive integers, find $m + n$.

    Solution: Let M be the foot of the perpendicular from O to BC. Then M is the midpoint of BC, so $DM = MB - DB = 2$. Let $FE = x$. We compute by Heron's formula that the area of $\triangle ABC$ is $\sqrt{21(6)(7)(8)} = 84$. Thus $AD = 168/14 = 12$. We can also compute the circumradius

    $$OB = \frac{(13)(14)(15)}{4 \cdot 84} = \frac{65}{8}.$$

    By the Pythagorean Theorem on OMB, we have

    $$OM = \sqrt{\frac{65^2}{8^2} - \frac{56^2}{8^2}} = \frac{\sqrt{(65-56)(65+56)}}{8} = \frac{\sqrt{9(121)}}{8} = \frac{33}{8}.$$

    Finally, we have $\triangle EMO \sim \triangle EDA$, so

    $$\frac{x}{x+2} = \frac{\frac{33}{8}}{12} = \frac{11}{32}$$

    so

    $$32x = 11x + 22$$

    or $x = \frac{22}{21}$. Thus, $DE = 2 + \frac{22}{21} = \frac{64}{21}$ and the answer is $\boxed{85}$.

    Proposed by Jerry Li

15. **Problem:** The set S consists of the first 10 positive integers. A collection of 10 not necessarily distinct integers is chosen from S at random. If a particular number is chosen more than once, all but one of its occurrences are removed. Call the set of remaining numbers A. Let $\frac{a}{b}$ be the expected value of the number of the elements in A, where a, b are relatively prime positive integers. Find the reminder when $a + b$ is divided by 1000.

Solution: We begin by finding the probability that a specific number k is chosen where $1 \le k \le 10$. The probability that k is the first number chosen is $\frac{1}{10}$, so the probability it is not the first number chosen is $1 - \frac{1}{10} = \frac{9}{10}$. Therefore, the probability that k is not chosen at all is $\left(\frac{9}{10}\right)^{10}$, so the probability it is chosen is $1 - \left(\frac{9}{10}\right)^{10}$. The expected value of the distinct number of chosen numbers is simply the sum of the probabilities each number from 1 to 10 is chosen, so the desired expected value is

$$10\left(1 - \left(\frac{9}{10}\right)^{10}\right) = \frac{10^{10} - 9^{10}}{10^9}.$$

We now want to find $10^{10} - 9^{10} + 10^9 \pmod{1000}$. The first and third terms are simply $0 \pmod{1000}$, so we are left with $-9^{10} \pmod{1000}$. By the Chinese Remainder Theorem, we can split this into determining $-9^{10} \pmod 8$ and $-9^{10} \pmod{125}$. The first expression simplifies to

$$-9^{10} \equiv -1^{10} \equiv -1 \equiv 7 \pmod 8.$$

The simplest way to evaluate the second expression is to find that $-9^2 \equiv 44 \pmod{125}$ and $-9^3 \equiv 21 \pmod{125}$ and multiplying these together to get

$$9^5 \equiv 49 \pmod{125} \Rightarrow -9^5 \equiv 76 \pmod{125}.$$

Squaring this gives $9^{10} \equiv 26 \pmod{125} \Rightarrow -9^{10} \equiv 99 \pmod{125}$. Combining our two congruences gives us $-9^{10} \equiv \boxed{599} \pmod{1000}$.

Proposed by Anuj Sakarda