Acton-Boxborough Math Competition Online Contest Solutions

Saturday, December 19 — Sunday, December 20, 2020

1. If $a \diamond b = ab - a + b$, find $(3 \diamond 4) \diamond 5$.

   Solution: Using order of operations, we know that we must solve the expression within the parentheses first. Replacing the variables with our given values, we get our first expression: 3*4 - 3+4. Solving, we get a value of 13. Then, we need to use this value for our second step of the expression. We can plug our given values once more into the expression and get: 13*5 - 13 + 5. Solving, we get 57. Our answer is $\boxed{57}$.

   Proposed by Jerry Li

2. If 5 chickens lay 5 eggs in 5 days, how many chickens are needed to lay 10 eggs in 10 days?

   Solution: We can count the number of eggs laid per chicken per day by the formula $\frac{\text{eggs}}{\text{chickens} \cdot \text{days}}$. We get the equation $\frac{5}{5 \cdot 5} = \frac{10}{x \cdot 10}$ where x is the number of chickens needed. Thus, we see that $\frac{1}{5} = \frac{1}{x}$ so $\boxed{5}$ chickens are needed.

   Proposed by Jerry Li

3. As Alissa left her house to go to work one hour away, she noticed that her odometer read 16261 miles. This number is a "special" number for Alissa because it is a palindrome and it contains exactly 1 prime digit. When she got home that evening, it had changed to the next greatest "special" number. What was Alissa's average speed, in miles per hour, during her two hour trip?

   Solution: First, we notice that the digit 2 is a prime number. This means that the hundredth's digit must change in order to only have 1 prime digit. After the number 16300, our next palindrome would be 16361. To find Alissa's average speed, we need to find the total distance travelled divided by the total amount of time it took her to travel. Her total distance travelled is 16361 - 16261 = 100 miles. And, we know her total time taken was two hours since she travelled a round trip of 1 hour + 1 hour. So, to calculate her average speed, we get: 100 miles/2 hours. Simplifying, we find her average speed to be 50 miles per hour. So, Alissa's average speed was $\boxed{50}$ mph.

   Proposed by Anusha Senapati

4. How many 1 in by 3 in by 8 in blocks can be placed in a 4 in by 4 in by 9 in box?

   Solution:

   Notice that the 8 in dimension of each block can only fit into the 9 in dimension of the box. Since all blocks must be placed to align the longest side to the height of the box, we can take a 2 dimensional cross-section (bird's eye view), and reduce this problem to "How many 1 inch by 3 inch rectangles can be placed in a 4 inch by 4 inch square?" Through a simple diagram we see that $\boxed{5}$ rectangles can fit in the square.

   Proposed by Anuj Sakarda

5. **Problem:** Apple loves eating bananas, but she prefers unripe ones. There are 12 bananas in each bunch sold. Given any bunch, if there is a $\frac{1}{3}$ probability that there are 4 ripe bananas, a $\frac{1}{6}$ probability that there are 6 ripe bananas, and a $\frac{1}{2}$ probability that there are 10 ripe bananas, what is the expected number of unripe bananas in 12 bunches of bananas?

   Solution: By definition of expected value we see that the expected number of ripe bananas in a bunch is $\frac{1}{3} \cdot 4 + \frac{1}{6} \cdot 6 + \frac{1}{2} \cdot 10 = \frac{4}{+}6$ ripe bananas and consequentially the expected number of unripe bananas in a bunch is $12 - \frac{4}{3} - 6 = \frac{14}{3}$. Therefore the expected number of unripe bananas in 12 bunches is $12 \cdot \frac{14}{3} = \boxed{56}$.

   Proposed by Annie Wang

6. **Problem:** The sum of the digits of a 3-digit number n is equal to the same number without the hundreds digit. What is the tens digit of n?

   Solution: Let then number be in the form $100a + 10b + c$. The sum of the digits of n is $a + b + c$ and the number without the hundreds digit is $10b + c$. Therefore we have the equation $a + b + c = 10b + c \Rightarrow a = 9b$. Therefore we see that $b = 1$ (otherwise the hundreds digit would exceed 10 or be zero, which is not allowed), so the tens digit is $\boxed{1}$.

   Proposed by Ethan Kuang

7. **Problem:** How many ordered pairs of positive integers (a, b) satisfy $a \le 20, b \le 20, ab > 15$?

   Solution: We use complementary counting for this question (so we count the number of $a, b \le 20$) such that $ab \le 15$ and subtract that from the total, or 400. We now proceed with casework:

   Case 1 : $a = 1$

   We can easily see that there are 15 values of b that work in this case (all positive integers from 1 to 15)

   Case 2 : $a = 2$

   We see that the values $1, 2, 3, 4, 5, 6, 7$ work, giving us 7 values for b .

   Case 3 : $a = 3$

   We see that the values $1, 2, 3, 4, 5$ work, giving us 5 values for b.

   Case 4 : $a = 4$

   We see that the values $1, 2, 3$ work, giving us 3 values for b.

   Case 5 : $a = 5$

   We see that the values $1, 2, 3$ work, giving us 3 values for b.

   Case 6 : $a = 6$

   We see that the values $1, 2$ work, giving us 2 values for b.

   Case 7 : $a = 7$

   We see that he values $1, 2$ work, giving us 2 values for b.

   Note that for $a \in \{8, 9, 10, 11, 12, 13, 14, 15\}$ the only value of b that works is 1, giving us 8 pairs.

   Therefore the total number of pairs (a, b) that satisfy $a, b \le 20$ and $ab \le 15$ is $15 + 7 + 5 + 3 + 3 + 2 + 2 + 8 = 45$. Hence, the total number of $a, b \le 20$ and $ab > 15$ is $400 - 45 = \boxed{355}$.

   Proposed by Jerry Li

8. **Problem:** Let $z(n)$ represent the number of trailing zeroes of $n!$. What is $z(z(6!))$? (Note: $n! = n \cdot (n-1) \cdots 2 \cdot 1$)

   Solution:

   We begin to find $z(6!)$. Since $z(n)$ is defined as the number of trailing zeros of $n!$ we are trying to find the number of trailing $0's$ of $720!$. Now we use Legendre's formula to calculate the number of $5's$ in the prime factorization (we can pair a 5 with a factor of 2 to create a factor of 10). Define $v_5(n)$ to be the number of factors of 5 in the prime factorization of n. We see

   $$v_5(720!) = \sum_{i=1} \left\lfloor \frac{720}{5^i} \right\rfloor = 144 + 28 + 5 + 1 = 178.$$

   Therefore $z(6!) = 178$.

   We now wish to find $z(178)$ or the number of trailing zero's after $178!$. We once again use Legendre's formula.

   $$v_5(178!) = \sum_{i=1} \left\lfloor \frac{178}{5^i} \right\rfloor = 35 + 7 + 1 = \boxed{43}.$$

Proposed by Devin Brown

9. **Problem:** On the Cartesian plane, points $A = (-1, 3), B = (1, 8)$ and $C = (0, 10)$ are marked. $\triangle ABC$ is reflected over the line $y = 2x + 3$ to obtain $\triangle A'B'C'$. The sum of the x-coordinates of the vertices of $\triangle A'B'C'$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$.

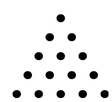
   Solution: Consider the lines through A, B and C and perpendicular to the line $y = 2x + 3$. These lines will have a slope of $-\frac{1}{2}$, and the points of intersections will be the midpoints of AA', BB', and CC' respectively.

   Thus we need to find the intersections of the lines $y - 3 = -\frac{1}{2}(x + 1)$, $y - 8 = -\frac{1}{2}(x - 1)$, and $y - 10 = -\frac{1}{2}x$ with $y = 2x + 3$.

   From this we can compute the midpoints $(-\frac{1}{5}, \frac{13}{5})$, $(\frac{11}{5}, \frac{37}{5})$, and $(\frac{14}{5}, \frac{43}{5})$ respectively, giving x coordinates of $\frac{3}{5}$, $\frac{17}{5}$, and $\frac{28}{5}$. Adding these gives us $\frac{48}{5}$, so the answer is $48 + 5 = \boxed{53}$.

   Proposed by Jerry Tan

10. How many ways can Bill pick three distinct points from the figure so that the points form a non-degenerate triangle?



   Solution: First consider the number of ways to pick three points from the picture, and then subtract cases where the three points do not form a triangle.

   We have $\binom{15}{3}$ total ways to pick three points, and the following cases that have 3 or more collinear points we must subtract from the total:

   Case 1: The three collinear points are on the largest side of the big triangle. We have $3 \cdot \binom{5}{3}$ ways that can occur

   Case 2: The three sets of four collinear points in the interior of the figure give $3 \cdot \binom{4}{3}$.

   Case 3: The three sides of the equilateral triangle at the center of the figure give 3.

   Case 4: The altitudes of the big triangle give 3 more.

   Thus we get $\binom{15}{3} - 3 \cdot \binom{5}{3} - 3 \cdot \binom{4}{3} - 3 - 3 = \boxed{407}$.

   Proposed by Devin Brown

11. **Problem:** Say piece A is attacking piece B if the piece B is on a square that piece A can move to. How many ways are there to place a king and a rook on an $8x8$ chessboard such that the rook isn't attacking the king, and the king isn't attacking the rook? Consider rotations of the board to be indistinguishable. (Note: rooks move horizontally or vertically by any number of squares, while kings move 1 square adjacent horizontally, vertically, or diagonally).

   Solution: We label each square using the standard chess board labelling, which labels the rows left to right as a through h and columns bottom to top as 1 through 8. The squares are then $a1, a2, a3, ...,$ $b1, b2, b3, ..., h1, h2, h3, h4, h5, h6, h7, h8$. We consider the following 3 cases:

   Case 1 : The King is in the corner

Since rotations are indistinguishable we only consider a King in one of the corners, say the a8 corner. The rook cannot be in the same row or column as the King, nor can it on the a8, b7 squares. Therefore in total the rook cannot be on $8 + 7 + 1 = 16$ squares so there are $64 - 16 = 48$ possible King-Rook configurations for this case.

Case 2 : The King is on an edge

Once again, since rotations are indistinguishable we only consider an edge, say the edge consisting of squares b8,c8,d8,e8,f8,g8. For each king placement there are exactly 17 squares that the rook cannot go on (for example if the King was on d8 then the rook cannot be on the 8th rank, nor can it be on the d-file, nor c7, e7 for 17 squares the rook cannot go to.). Therefore there is a total of $6(64 - 17) = 282$ possible King-Rook configurations.

Case 3 : The King is anywhere else.

Now we consider a quadrant of the board. We cannot have the king in one of the edges (nor corner), as we have already accounted for that case above. Therefore there are $16 - 7 = 9$ possible squares the king be on. For each King square, there are 19 squares the rook isn't allowed to go. Therefore the total number possible of King-Rook configurations for this case is $9(64 - 19) = 405$.

Hence summing up our cases we see that there are $405 + 282 + 48 = \boxed{735}$ King-rook configurations.

Proposed by Jerry Li

12. **Problem:** Let the remainder when $P(x) = x^{2020} - x^{2017} - 1$ is divided by $S(x) = x^3 - 7$ be the polynomial $R(x) = ax^2 + bx + c$ for integers a, b, c. Find the remainder when $R(1)$ is divided by 1000.

Solution: Let $P(x) = S(x)Q(x) + R(x)$ where $Q(x)$ has integer coefficients. Notice that if we let $x^3 = 7$, then $P(x) = 0 + R(x) = R(x)$. This means to find $R(x)$, we replace x^3 with 7 in $P(x)$ repeatedly. We can write

$$P(x) = (x^3)^{673}x - (x^3)^{672}x - 1,$$

so $R(x) = 7^{673}x - 7^{672}x - 1$. Then $R(1) = 7^{673} - 7^{672} - 1$.

It would help to find the order of 7 mod(1000), or the smallest positive integer exponent d with $7^d \equiv 1 \pmod{1000}$. We notice that $7^4 = 2401 = 24(100) + 1 \equiv 1 \pmod{100}$. Then d must be a multiple of 4. Then

$$(7^4)^c \equiv c \cdot 24(100) + 1 \pmod{1000}$$

by the Binomial Theorem. If $c = 5$, then $7^{20} \equiv 1 \pmod{1000}$. Now $R(1) \equiv 7^{13} - 7^{12} - 1 \equiv 407 - 201 - 1 \equiv \boxed{205} \pmod{1000}$.

Proposed by Jerry Tan

13. **Problem:** Let $S(x) = \left\lfloor \frac{2020}{x} \right\rfloor + \left\lfloor \frac{2020}{x+1} \right\rfloor$. Find the number of distinct values $S(x)$ achieves for integers x in the interval $[1, 2020]$.

Solution: Denote $f(x) = \left\lfloor \frac{2020}{x} \right\rfloor$. Let x_0 be the smallest x such that $\frac{2020}{x} - \frac{2020}{x+1} < \frac{1}{2}$. Knowing x_0 is useful because for x smaller than x_0, $f(x-1)$ and $f(x+1)$ are guaranteed to be different, since $\frac{2020}{x-1} - \frac{2020}{x+1} > 2 \cdot \frac{1}{2} = 1$. This means that $S(x-1) \neq S(x)$ for $x < x_0$. So, for $1 \leq x < x_0$, each $S(x)$ takes on a distinct value.

On the other hand, for $x \geq x_0$ since $\frac{2020}{x} - \frac{2020}{x+2} < 1$, we know $S(x)$ and $S(x+1)$ cannot differ by more than 1. This means that for $x_0 \leq x \leq 2020$, the function $S(x)$ will take on all integer values between $S(x_0)$ and $S(2020)$ without skipping any integers in between.

Now we must find x_0. Solving we get

$$\begin{aligned} \frac{2020}{x} - \frac{2020}{x+1} &< \frac{1}{2} \\ 4040(x+1) - 4040x &< x(x+1) \\ 0 &< x^2 + x - 4040 \\ 64 &\leq x, \end{aligned}$$

since we are only concerned about integers x. Using $x_0 = 64$ and the work above, we know $S(x)$ takes on 63 distinct values for $1 \leq x < x_0 = 64$, and $S(x)$ must take on all integers between $S(64) = 62$ and $S(2020) = 1$, or a total of 62 values in this case. The total is then $63 + 62 = \boxed{125}$ distinct values.

Proposed by Jerry Tan

14. **Problem:** Triangle $\triangle ABC$ is inscribed in a circle with center O and has sides $AB = 24, BC = 25, CA = 26$. Let M be the midpoint of $\overline{AB}$. Points K and L are chosen on sides $\overline{BC}$ and $\overline{CA}$, respectively such that $BK < KC$ and $CL < LA$. Given that $OM = OL = OK$, the area of triangle $\triangle MLK$ can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers, $\gcd(a, c) = 1$ and b is not divisible by the square of any prime. Find $a + b + c$.

Solution: The condition that $OM = OL = OK$ means that M, L, K have equal power with respect to the circumcircle of $\triangle ABC$. Thus,

$$144 = MA \cdot MB = KB \cdot KC = LC \cdot LA.$$

Since $KB + KC = 25$ and $LC + LA = 26$, we can solve to get $KB = 9, KC = 16, LC = 8, LB = 18$.

Let $[ABC]$ denote the area of triangle $\triangle ABC$. By Heron's,

$$[ABC] = \sqrt{\frac{75}{2} \cdot \frac{23}{2} \cdot \frac{25}{2} \cdot \frac{27}{2}} = \frac{225\sqrt{23}}{4}.$$

Then,

$$\frac{[MLA]}{[ABC]} = \frac{MA}{AB} \cdot \frac{LA}{AC} = \frac{12}{24} \cdot \frac{18}{26} \to [MLA] = \frac{9}{26}[ABC].$$

Similarly,

$$[LCK] = \frac{8 \cdot 16}{26 \cdot 25}[ABC] = \frac{64}{325}[ABC], \quad [KBM] = \frac{9 \cdot 12}{25 \cdot 24}[ABC] = \frac{9}{50}[ABC].$$

Finally,

$$[MLK] = [ABC] - [MLA] - [LCK] - [KBM] = [ABC]\left(\frac{650 - 225 - 128 - 117}{650}\right) = \frac{180}{650}[ABC] = \frac{405\sqrt{23}}{26}.$$

The final answer is then $405 + 23 + 26 = \boxed{454}$.

Proposed by Jerry Tan

15. **Problem:** Euler's totient function, $\phi(n)$, is defined as the number of positive integers less than n that are relatively prime to n. Let $S(n)$ be the set of composite divisors of n. Evaluate

$$\sum_{k=1}^{50} \left(k - \sum_{d \in S(k)} \phi(d) \right).$$

Solution: We use the following well-known identity:

$$\sum_{d|n} \phi(n) = n.$$

Then

$$\sum_{d \in S(k)} \phi(d) = k - P(k) - 1$$

where $P(k)$ is the sum of the totient function of the prime divisors of k. Thus,

$$\sum_{k=1}^{50} \left(k - \sum_{d \in S(k)} \phi(d) \right) = 50 + \sum_{k=1}^{50} P(k).$$

To calculate the sum on the right, notice that $\phi(p)$ contributes to the sum a total of $\left\lfloor \frac{50}{p} \right\rfloor$ times, once for each multiple of p less than or equal to 50. So,

$$\begin{aligned}\sum_{k=1}^{50} P(k) &= 25\phi(2) + 16\phi(3) + 10\phi(5) + 7\phi(7) + 4\phi(11) + 3\phi(13) + 2\phi(17) + 2\phi(19) \\ &\quad + 2\phi(23) + \phi(29) + \phi(31) + \phi(37) + \phi(41) + \phi(43) + \phi(47) \\ &= 25 + 32 + 40 + 42 + 40 + 36 + 32 + 36 + 44 + 28 + 30 + 36 + 40 + 42 + 46 \\ &= 549.\end{aligned}$$

Therefore, $50 + \sum_{k=1}^{50} P(k) = \boxed{599}$.

Proposed by Anuj Sakarda