Acton-Boxborough Math Competition Online Contest Solutions

Saturday, November 18 — Sunday, November 19, 2023

1. **Problem:** There are 2024 apples in a very large basket. First, Julie takes away half of the apples in the basket; then, Diane takes away 202 apples from the remaining bunch. How many apples remain in the basket?

   Solution: (Eric Li) There are 2024 apples in a basket and that Julie takes half of the apples in the basket. This means that Julie takes away 1012 apples, so there are 1012 apples left. Diane takes away 202 apples, which means that there will be 1012 - 202 = $\boxed{810}$ apples remaining.

   Proposed by Ivy Shi

2. **Problem:** The set of all permutations (different arrangements) of the letters in "ABMC" are listed in alphabetical order. The first item on the list is numbered 1, the second item is numbered 2, and in general, the kth item on the list is numbered k. What number is given to "ABMC"?

   Solution: (Daniel Ren) Listing out the first few terms:

   $$\text{ABCM, ABMC, ACBM, } \cdots$$

   We see that ABMC is term $\boxed{2}$ in this list.

   Proposed by Daniel Cai

3. **Problem:** Daniel has a water bottle that is three-quarters full. After drinking 3 ounces of water, the water bottle is three-fifths full. The density of water is 1 gram per milliliter, and there are around 28 grams per ounce. How many milliliters of water could the bottle fit at full capacity?

   Solution: (Bryan Li) The fraction of the total water that Daniel drinks is $3/4 - 3/5 = 3/20$, and he has drank $3 \cdot 28 = 84$ milliliters. If x is the amount of water the water bottle fits, we have $\frac{3}{20} = \frac{84}{x}$. Solving, we obtain $x = 28 \cdot 20 = \boxed{560}$ milliliters.

   Proposed by Raymond Gao

4. **Problem:** How many ways can four distinct 2-by-1 rectangles fit on a 2-by-4 board such that each rectangle is fully on the board?

   Solution: (Iris Shi) It is impossible for the board to be filled with an odd number of horizontally-oriented rectangles (long side horizontal) or vertically-oriented rectangles (long side vertical). The number of ways to shuffle the rectangles is $4! = 24$. There are $24 \cdot 1$ ways to place four rectangles vertically, $24 \cdot 3$ ways to place two rectangles vertically and two rectangles horizontally, and $24 \cdot 1$ ways to place four rectangles horizontally. Overall, there are $24 \cdot 1 + 24 \cdot 3 + 24 \cdot 1 = 24 \cdot 5 = \boxed{120}$ ways to arrange the rectangles.

   Proposed by Iris Shi

5. **Problem:** Iris and Ivy start reading a 240 page textbook with 120 left-hand pages and 120 right-hand pages. Iris takes 4 minutes to read each page, while Ivy takes 5 minutes to read a left-hand page and 3 minutes to read a right-hand page.

   Iris and Ivy move onto the next page only when both sisters have completed reading. If a sister finishes reading a page first, the other sister will start reading three times as fast until she completes the page.

   How many minutes after they start reading will both sisters finish the textbook?

   Solution: (Iris Shi) For left-hand pages, Iris will finish reading in 4 minutes while Ivy will still have 1 minute of reading left. In this case, Ivy will start reading at three times the speed and finish reading after $1/3$ minutes, totaling $13/3$ minutes for each left-hand page. For right-hand pages, Ivy will finish reading in 3 minutes while Iris will still have 1 minute of reading left. Then, Iris will start reading at three times the speed and finish reading after $1/3$ minutes, totaling $10/3$ minutes for each right-hand page. The total time it takes for Iris and Ivy to finish the book is $120 \cdot 13/3 + 120 \cdot 10/3 = \boxed{920}$ minutes.

Proposed by Daniel Ren

6. **Problem:** Let $\triangle ABC$ be an equilateral triangle with side length 24. Then, let M be the midpoint of BC. Define $\mathcal{P}$ to be the set of all points P such that $2PM = BC$. The minimum value of AP can be expressed as $\sqrt{a} - b$, where a and b are positive integers. Find $a + b$.

   Solution: (Iris Shi) From the information given, we see that the set $\mathcal{P}$ is the circle with diameter BC and center M. The closest point P to A is the intersection of $\mathcal{P}$ and AM. The altitude AM of the equilateral triangle can be found with 30-60-90 triangle ratios; we have $AD = CD\sqrt{3} = 12\sqrt{3}$. Also, $PM = BC/2 = 12$. Since $AP = AM - PM$, we have

$$AP = 12\sqrt{3} - 12 = \sqrt{432} - 12.$$

   The final answer is $432 + 12 = \boxed{444}$.

   Proposed by Jerry Li

7. **Problem:** Jonathan has 10 songs in his playlist: 4 rap songs and 6 pop songs. He will select three unique songs to listen to while he studies. Let p be the probability that at least two songs are rap, and let q be the probability that none of them are rap. Find $\frac{p}{q}$.

   Solution: (Daniel Ren) There are $\binom{10}{3}$ ways that Jonathan can pick three songs out of a pool of ten.

   The number of ways Jonathan can pick *at least* two rap songs can be split into cases:

   1: He picks two rap songs and one pop song. He can select one out of 6 pop songs, then $\binom{4}{2} = 6$ pairs of rap songs.

   2: He picks three rap songs. This can happen in $\binom{4}{3} = 4$ ways.

   In all, there are $6 \cdot 6 + 4 = 40$ ways Jonathan can pick at least two rap songs.

   The number of ways Jonathan can pick no rap songs and three pop songs is $\binom{6}{3} = 20$ ways.

   The ratio p/q is

$$\frac{40/\binom{10}{3}}{20/\binom{10}{3}} = \frac{40}{20} = \boxed{2}.$$

   Proposed by Raymond Gao

8. **Problem:** A number K is called *6,8-similar* if K written in base 6 and K written in base 8 have the same number of digits. Find the number of 6,8-similar values between 1 and 1000, inclusive.

   Solution: (Daniel Ren) We split into cases depending on the number of digits K has in base 6 and base 8.

   b6: When $1 \leq K \leq 5$, K has one digit in base 6 (from 1_6 to 5_6).
   When $6 \leq K \leq 35$, K has two digits in base 6 (from 10_6 to 55_6).
   When $36 \leq K \leq 215$, K has three digits in base 6 (from 100_6 to 555_6).
   When $216 \leq K \leq 1000$, K has four digits in base 6 (from 1000_6 to 4344_6).

   b8: When $1 \leq K \leq 7$, K has one digit in base 8 (from 1_8 to 7_8).
   When $8 \leq K \leq 63$, K has two digits in base 8 (from 10_8 to 77_8).
   When $64 \leq K \leq 511$, K has three digits in base 8 (from 100_8 to 77_8).
   When $512 \leq K \leq 1000$, K has four digits in base 8 (from 1000_8 to 1750_8).

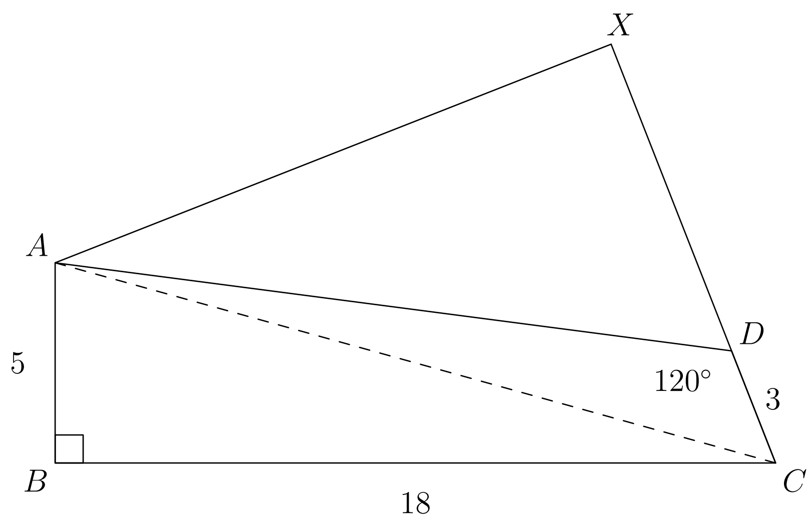
The numbers between 1 to 5, 8 to 35, 64 to 215, and 512 to 1000 have the same number of digits when written in base 6 and base 8. In total, there are $5 + 28 + 152 + 489 = \boxed{674}$ numbers below 1000 with this property.

Note: X_b denotes the number X in base b. For example, $35_6 = 23_{10}$. When writing numbers in base 10, the $_{10}$ subscript is usually omitted.

Proposed by Daniel Ren

9. **Problem:** Quadrilateral $ABCD$ has $\angle ABC = 90^\circ$, $\angle ADC = 120^\circ$, $AB = 5$, $BC = 18$, and $CD = 3$. Find AD^2.

   Solution: (Daniel Ren) Extend $\overline{CD}$. Then, draw the altitude from A to $\overleftrightarrow{CD}$, letting the foot of the altitude be X.



   We have $\angle AXD = 90^\circ$ and $\angle ADX = 180^\circ - 120^\circ = 60^\circ$, so $\triangle AXD$ is a 30-60-90 triangle.

   Let $XD = x$. By 30-60-90 triangle ratios, we know that $AX = \sqrt{3}x$ and $AD = 2x$. Now, we can express AC in two ways with the Pythagorean theorem:

$$\begin{aligned} AC^2 = 18^2 + 5^2 &= XA^2 + XC^2 \\ 349 &= (\sqrt{3}x)^2 + (x+3)^2 \\ &= 4x^2 + 6x + 9. \end{aligned}$$

   Moving all terms to one side and simplifying, we obtain the quadratic $2x^2 + 3x - 170 = 0$. Factoring into $(2x - 17)(x + 10)$, we find that $x = 17/2$.

   Finally, $AD^2 = (2x)^2 = 17^2 = \boxed{289}$.

   Proposed by Raymond Gao

10. **Problem:** Bob, Eric, and Raymond are playing a game. Each player rolls a fair 6-sided die, and whoever has the highest roll wins. If players are tied for the highest roll, the ones that are tied reroll until one wins. At the start, Bob rolls a 4. The probability that Eric wins the game can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

    Solution: (Bryan Li) Let us first compute the probability that Bob wins. Consider 3 cases:

    1: Bob wins on the first turn. This happens if Eric and Raymond both roll below 4, which happens with $1/2 \cdot 1/2 = 1/4$th of the time.

2: Bob ties with Eric on the first turn. This happens if either Eric rolls a 4, while Raymond rolls below a 4. Then, Bob has a $1/2$ chance of winning after that, for a total probability of $1/6 \cdot 1/2 \cdot 1/2 = 1/24$.

3: Bob ties with Raymond on the first turn. This case is the same as the last one, and Bob wins with probability $1/24$.

4: Everyone ties. This happens when both Eric and Raymond roll a 4; Bob has a $1/3$rd chance of winning after that for a total probability of $1/6 \cdot 1/6 \cdot 1/3 = 1/108$.

Therefore, Bob wins with a $1/4 + 1/24 + 1/108 = 37/108$ probability. The probability that he doesn't win is $1 - 37/108 = 71/108$. Out of that, Eric and Raymond win with equal probability, so Eric wins with probability $71/216$. Our final answer is $71 + 216 = \boxed{287}$.

Proposed by Daniel Cai

11. **Problem:** Define the following infinite sequence s:

$$s = \left\{ \frac{9}{2}, \frac{99}{2^2}, \frac{999}{2^3}, \dots, \frac{\overbrace{999...999}^{k \text{ nines}}}{2^k}, \dots \right\}$$

The sum of the first 2024 terms in s, denoted S, can be expressed as

$$S = \frac{5^a - b}{4} + \frac{1}{2^c},$$

where a, b, and c are positive integers. Find $a + b + c$.

Solution: (Ben Zhu) Examining the individual terms in s, we notice that $\overbrace{999...999}^{k \text{ nines}} = 10^k - 1$. We can write a more general expression for the kth term in s:

$$s_k = \frac{10^k - 1}{2^k} = \frac{10^k}{2^k} - \frac{1}{2^k} = 5^k - \frac{1}{2^k}.$$

Knowing this, we can separate S into the sum of two geometric series:

$$S = \sum_{k=1}^{2024} 5^k - \sum_{k=1}^{2024} \frac{1}{2^k}.$$

The closed form expression of a finite geometric series can be expressed as follows:

$$\sum_{k=1}^{n} ar^k = \frac{a(r^n - 1)}{r - 1}.$$

Applying this formula twice to both geometric series, we obtain

$$\begin{aligned} S &= \frac{5(5^{2024} - 1)}{5 - 1} - \frac{\frac{1}{2}\left(\frac{1}{2^{2024}} - 1\right)}{\frac{1}{2} - 1} \\ &= \frac{5^{2025} - 5}{4} + \frac{1}{2^{2024}} - 1 \\ &= \frac{5^{2025} - 9}{4} + \frac{1}{2^{2024}}. \end{aligned}$$

Our final answer is $a + b + c = 2025 + 9 + 2024 = \boxed{4058}$.

Proposed by Daniel Ren

12. **Problem:** Andy is adding numbers in base 5. However, he accidentally forgets to write the units digit of each number. If he writes all the consecutive integers starting at 0 and ending at 50 (base 10) and adds them together, what is the difference between Andy's sum and the correct sum? (Express your answer in base-10.)

    Solution: (Eric Xiang) The correct sum (in base 10) is $\frac{1}{2} \cdot 50 \cdot 51 = 1275$. The incorrect sum is obtained as follows: The base-5 representation of 50 is 200_5, so we only have to consider one, two, and three-digit numbers.

    The single-digit numbers count as 0 after getting rid of their units digits, so we can disregard them.

    Two-digit numbers, after getting of their units digits, have only their "fives" digit left. Now, take two-digit numbers starting with 1 for example. There are 5 possible digits for the ones digit (0, 1, 2, 3, 4), so when we add these numbers without their units digits, we will add 1 five times. This works similarly for two-digit numbers beginning with other "fives" digits, and the three-digit numbers as well, except that we add the number formed by the first two digits five times – there are 5 numbers beginning with $10, 11, 12, 13$, and 14. There is one number beginning with 20, which is $200_5 = 50_10$

    We are simply adding the numbers 1 to 14 in base 5, five times, then adding a 20. This is equivalent to adding from 1 through 9, multiplying by 5, then adding 10 in base 10. Hence, the wrong sum is $45 \cdot 5 + 10 = 235$.

    Finally, the difference between the correct and incorrect sums is $1275 - 235 = \boxed{1040}$.

    Proposed by Daniel Cai

13. **Problem:** (Daniel Ren) Let n be the positive real number such that the system of equations

$$y = \frac{1}{\sqrt{2024 - x^2}}$$
$$y = \sqrt{x^2 - n}$$

has exactly two real solutions for (x, y): (a, b) and $(-a, b)$. Then, $|a|$ can be expressed as $j\sqrt{k}$, where j and k are integers such that k is not divisible by any perfect square other than 1. Find $j \cdot k$.

Solution: (Daniel Ren) To solve, we set the two equations equal to each other:

$$\sqrt{x^2 - n} = \frac{1}{\sqrt{2024 - x^2}}$$
$$\sqrt{(x^2 - n)(2024 - x^2)} = 1$$
$$-(x^4 - (2024 + n)x^2 + 2024n) = 1 \qquad (*)$$

This quartic has exactly two solutions. Let $z = x^2$, so $x = \sqrt{z}$ or $-\sqrt{z}$. Rewriting the above equation in terms of z should yield a quadratic with one solution:

$$z^2 - (2024 + n)z + (2024n + 1) = 0.$$

In order for a general quadratic $c_2x^2 + c_1x + c_0 = 0$ to have one solution, c_0 must be equal to $(c_1/2)^2$. We can use this property to solve for n:

$$2024n + 1 = \left(\frac{2024 + n}{2}\right)^2$$
$$4(2024n + 1) = (2024 + n)^2$$
$$4(2024n) + 4 = n^2 + 2(2024n) + 2024^2.$$

Moving all terms to one side, we get the quadratic:

$$n^2 - 2(2024)n + (2024^2 - 4) = 0.$$

Using difference of squares, we may write $2024^2 - 4$ as $(2022)(2026)$. Then, we can easily factor the quadratic:

$$(n-2022)(n-2026) = 0$$
$$n = 2022, 2026.$$

When plugging back into original system, 2026 is an extraneous solution (since x^2 must be less than 2024 from the second equation). Therefore, n can only equal 2022. Plugging this back into equation $(*)$, we can solve for x:

$$\begin{aligned} x^4 - (2024+2022)x^2 + (2024)(2022) + 1 &= 0 \\ x^4 - 2(2023)x^2 + 2023^2 &= 0 \\ (x^2 - 2023)^2 &= 0 \\ x &= \pm\sqrt{2023} = \pm 17\sqrt{7}. \end{aligned}$$

Our final answer is $17 \cdot 7 = \boxed{119}$.

Proposed by Daniel Ren

14. **Problem:** Nakio is playing a game with three fair 4-sided dice. But being the cheater he is, he has secretly replaced one of the three die with his own 4-sided die, such that there is a $1/2$ chance of rolling a 4, and a $1/6$ chance to roll each number from 1 to 3. To play, a random die is chosen with equal probability and rolled. If Nakio guesses the number that is on the die, he wins.

Unfortunately for him, Nakio's friends have an anti-cheating mechanism in place: when the die is picked, they will roll it three times. If each roll lands on the same number, that die is thrown out and one of the two unused dice is chosen instead with equal probability.

If Nakio always guesses 4, the probability that he wins the game can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Find $m+n$.

Solution: (Bryan Li, Jason Liu) Let the probability that the friends use the weighted die be p. Then, Nakio's die can either be used with probability p (so his chances of winning are $1/2$), or a normal die can be used with probability $1-p$ (with a $1/4$th chance that he wins). The probability Nakio wins, expressed in terms of p, is

$$\frac{1}{2}p + \frac{1}{4}(1-p) = \frac{1}{4}p + \frac{1}{4}.$$

The weighted die will be picked in one of two ways: either it is chosen first and passes the anti-cheating method, or it is chosen second after a normal die fails the anti-cheating method. The first way happens with probability

$$\frac{1}{3}\left(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + 3\cdot\frac{1}{6}\cdot\frac{5}{6} + 3\cdot\frac{1}{6}\cdot\frac{5}{6}\right) = \frac{31}{108},$$

and the second way happens with probability

$$\frac{2}{3}\cdot\frac{1}{2}\left(4\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right) = \frac{1}{48}.$$

The probability is

$$\begin{aligned} \frac{m}{n} &= \frac{1}{4}p + \frac{1}{4} = \frac{1}{4}\left(\frac{31}{108} + \frac{1}{48}\right) + \frac{1}{4} \\ &= \frac{133}{1728} + \frac{1}{4} = \frac{565}{1728}. \end{aligned}$$

The final answer is $565 + 1728 = \boxed{2293}$.

Proposed by Bryan Li

15. **Problem:** A particle starts in the center of a 2m-by-2m square. It moves in a random direction such that the angle between its direction and a side of the square is a multiple of 30°. It travels in that direction at 1m/s, bouncing off of the walls of the square. After a minute, the position of the particle is recorded. The expected distance from this point to the start point can be written as

$$\frac{1}{a}\left(b - c\sqrt{d}\right),$$

where a and b are relatively prime, and d is not divisible by any perfect square. Find $a + b + c + d$.

Solution: (Bryan Li) Notice that we can reflect the square across its edges to make an infinite grid of 2m-by-2m squares. Then, set the starting point to $(0, 0)$. Abstractly, the original square's starting position is duplicated infinitely, and the particle reflecting off of walls is analogous to the particle passing through the walls.

In our new problem, the particle now radiates from the origin for 60 seconds, becoming a distance of 60 meters away from it. The particle's starting positions are all points on the Cartesian Plane with even-integer coordinates.

We can consider only two cases: where the particle radiates 0° with respect to the positive x-axis, and where the particle radiates 30° with respect to the positive x-axis. The 60° case is analogous to the second case, and the rest of the cases can be ignored due to symmetry.

1: 0°. The particle will end at the point $(60, 0)$. This point is exactly on a starting point, so its distance away from it is 0. There are four angles (0°, 90°, 180°, 270°)

2: 30°. The particle will end at the point $(30\sqrt{3}, 30)$. The closest starting point is $(52, 30)$, and the distance between these two points is $52 - 30\sqrt{3}$. There are eight angles (30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°) which are these distances away.

The expected distance is

$$\frac{1}{12}\left(4(0) + 8(52 - 30\sqrt{3})\right) = \frac{1}{12}\left(416 - 240\sqrt{3}\right) = \frac{1}{3}\left(104 - 60\sqrt{3}\right).$$

Our final answer is $3 + 104 + 60 + 3 = \boxed{170}$.

Proposed by Bryan Li