Acton-Boxborough Math Competition Online Contest

Saturday, October 21 — Sunday, October 22, 2023

Contest Rules and Format

The 2023 October Contest consists of 15 problems worth 1 point each — each with an answer between 0 and 100,000. The contest window is

Saturday, October 21 to Sunday, October 22.

The following rules supersede any others regarding the Online Contests.

- To submit your responses, fill out a form on our website, `abmathcompetition.wordpress.com`. Include your email, name, school, grade (we require this information for prize purposes).
- Submissions will close at 10:00PM Eastern Standard Time, on the final day of the round.
- Computational aids such as WolframAlpha, calculators, abaci, and other related materials are prohibited. Graphing technology such as GeoGebra, Desmos, or a graphing calculator are also prohibited. Drawing aids such as rulers and protractors are permissible.
- You may only work individually on the problems—consultation with others is not permitted.
- You may take as much time as you wish during the contest window.
- Ties will be broken by the most difficult problem solved, determined by the number of correct answers to the question. If problem A is solved by a contestants, and problem B is solved by b contestants, with $a < b$, then problem A is said to be "more difficult" than B.

Awards and Prizes

- Top scoring contestants within each round will have their names posted on our website.
- Scores will be emailed to contestants within one week of the contest's end.
- Top scoring contestants across all three rounds will receive prizes.
- Physical prizes (such as AoPS gift certificates, merchandise, calculators, etc.) will be given to top scorers if they attend the ABMC onsite contest.

Thanks to our Sponsors!

Good luck!

Problems

1. What is $2 \cdot 24 + 20 \cdot 24 + 202 \cdot 4 + 2024$?

2. Jerry has 300 legos. He can either make cars, which require 17 legos, or bikes, which require 13 legos. Assuming he uses all of his legos, how many ordered pairs (a, b) are there such that he makes a cars and b bikes?

3. Patrick has 7 unique textbooks: 2 Geometry books, 3 Precalculus books and 2 Algebra II books. How many ways can he arrange his books on a bookshelf such that all the books of the same subjects are adjacent to each other?

4. After a hurricane, a 32 meter tall flagpole at the Acton-Boxborough Regional High School snapped and fell over. Given that the snapped part remains in contact with the original pole, and the top of the pole falls 24 meters away from the bottom of the pole, at which height did the pole snap? (Assume the flagpole is perpendicular to the ground.)

5. Jimmy is selling lemonade. He has 200 cups of lemonade, and he will sell them all by the end of the day. Being the ethically dubious individual he is, Jimmy intends to dilute a few of the cups of lemonade with water to conserve resources.

   Jimmy sells each cup for \$4. It costs him \$1 to make a diluted cup of lemonade, and it costs him \$2.75 to make a cup of normal lemonade. What is the minimum number of diluted cups Jimmy must sell to make a profit of over \$400?

6. Jeffrey has a bag filled with five fair dice: one with 4 sides, one with 6 sides, one with 8 sides, one with 12 sides, and one with 20 sides. The dice are numbered from 1 to the number of sides on the die. Now, Marco will randomly pick a die from Jeffrey's bag and roll it. The probability that Marco rolls a 7 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

7. What is the remainder when the sum of the first 2024 odd numbers is divided by 6072?

8. A rhombus $ABCD$ with $\angle A = 60^\circ$ and $AB = 600$ cm is drawn on a piece of paper. Three ants start moving from point A to the three other points on the rhombus.

   One ant walks from A to B at a leisurely speed of 10 cm/s. The second ant runs from A to C at a slightly quicker pace of $6\sqrt{3}$ cm/s, arriving to C x seconds after the first ant. The third ant travels from A to B to D at a constant speed, arriving at D x seconds after the second ant.

   The speed of the last ant can be written as $\frac{m}{n}$ cm/s, where m and n are relatively prime positive integers. Find mn.

9. This year, the Apple family has harvested so many apples that they cannot sell them all! Applejack decides to make 40 glasses of apple cider to give to her friends. If Twilight and Fluttershy each want 1 or 2 glasses; Pinkie Pie wants either 2, 14, or 15 glasses; Rarity wants an amount of glasses that is a power of three; and Rainbow Dash wants any odd number of glasses, then how many ways can Applejack give her apple cider to her friends?

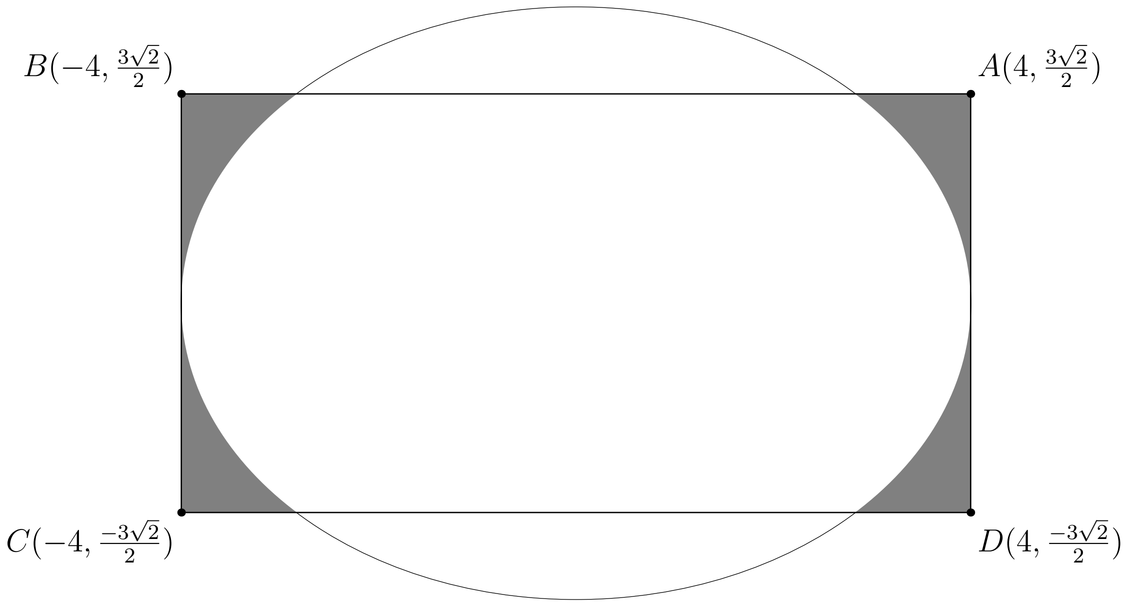
   Note: 1 is considered to be a power of 3.

10. Let g_x be a geometric sequence with first term 27 and successive ratio $2n$ (so $g_{x+1}/g_x = 2n$). Then, define a function f as $f(x) = \log_n(g_x)$, where n is the base of the logarithm. It is known that the sum of the first seven terms of $f(x)$ is 42. Find g_2, the second term of the geometric sequence.

    Note: The *logarithm* base b of x, denoted $\log_b(x)$, is equal to the value y such that $b^y = x$. In other words, if $\log_b(x) = y$, then $b^y = x$.

11. Let $\mathcal{E}$ be an ellipse centered around the origin, such that its minor axis is perpendicular to the x-axis. The length of the ellipse's major and minor axes is 8 and 6, respectively. Then, let $ABCD$ be a rectangle centered around the origin, such that AB is parallel to the x-axis. The lengths of AB and BC are 8 and $3\sqrt{2}$, respectively.

    The area outside the ellipse but inside the rectangle can be expressed as $a\sqrt{b} - c - d\pi$, for position integers a, b, c, d where b is not divisible by a perfect square of any prime. Find $a + b + c + d$.



12. Let $N = 2^7 \cdot 3^7 \cdot 5^5$. Find the number of ways to express N as the product of squares and cubes, all of which are integers greater than 1.

13. Jerry and Eric are playing a 10-card game where Jerry is deemed the "landlord" and Eric is deemed the "peasant".

    To deal the cards, the landlord keeps one card to himself. Then, the rest of the 9 cards are dealt out, such that each card has a 1/2 chance to go to each player. Once all 10 cards are dealt out, the landlord compares the number of cards he owns with his peasant. The probability that the landlord wins is the fraction of cards he has. (For example, if Jerry has 5 cards and Eric has 2 cards, Jerry has a 5/7ths chance of winning.)

    The probability that Jerry wins the game can be written as $\frac{p}{q}$, where p and q are relatively prime. Find $p + q$.

14. Define $P(x) = 20x^4 + 24x^3 + 10x^2 + 21x + 7$ to have roots a, b, c, and d. If $Q(x)$ has roots $\frac{1}{a-2}$, $\frac{1}{b-2}$, $\frac{1}{c-2}$, $\frac{1}{d-2}$, and integer coefficients with a greatest common divisor of 1, then find $Q(2)$.

15. Let $\triangle ABC$ be a triangle with side lengths $AB = 14$, $BC = 13$, and $AC = 15$. The incircle of $\triangle ABC$ is drawn with center I, tangent to $\overline{AB}$ at X. The line $\overleftrightarrow{IX}$ intersects the incircle again at Y and intersects $\overline{AC}$ at Z. The area of $\triangle AYZ$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.