Acton-Boxborough Math Competition Online Contest

Saturday, November 18 — Sunday, November 19, 2023

Contest Rules and Format

The 2023 November Contest consists of 15 problems worth 1 point each — each with an answer between 0 and 100,000. The contest window is

Saturday, November 18 to Sunday, November 19.

The following rules supersede any others regarding the Online Contests.

- To submit your responses, fill out a form on our website, `abmathcompetition.wordpress.com`. Include your email, name, school, grade (we require this information for prize purposes).
- Submissions will close at 10:00PM Eastern Standard Time, on the final day of the round.
- Computational aids such as WolframAlpha, calculators, abaci, and other related materials are prohibited. Graphing technology such as GeoGebra, Desmos, or a graphing calculator are also prohibited. Drawing aids such as rulers and protractors are permissible.
- You may only work individually on the problems—consultation with others is not permitted.
- You may take as much time as you wish during the contest window.
- Ties will be broken by the most difficult problem solved, determined by the number of correct answers to the question. If problem A is solved by a contestants, and problem B is solved by b contestants, with $a < b$, then problem A is said to be "more difficult" than B.

Awards and Prizes

- Top scoring contestants within each round will have their names posted on our website.
- Scores will be emailed to contestants within one week of the contest's end.
- Top scoring contestants across all three rounds will receive prizes.
- Physical prizes (such as AoPS gift certificates, merchandise, calculators, etc.) will be given to top scorers if they attend the ABMC onsite contest.

Thanks to our Sponsors!

Good luck!

Problems

1. There are 2024 apples in a very large basket. First, Julie takes away half of the apples in the basket; then, Diane takes away 202 apples from the remaining bunch. How many apples remain in the basket?

2. The set of all permutations (different arrangements) of the letters in "ABMC" are listed in alphabetical order. The first item on the list is numbered 1, the second item is numbered 2, and in general, the kth item on the list is numbered k. What number is given to "ABMC"?

3. Daniel has a water bottle that is three-quarters full. After drinking 3 ounces of water, the water bottle is three-fifths full. The density of water is 1 gram per milliliter, and there are around 28 grams per ounce. How many milliliters of water could the bottle fit at full capacity?

4. How many ways can four distinct 2-by-1 rectangles fit on a 2-by-4 board such that each rectangle is fully on the board?

5. Iris and Ivy start reading a 240 page textbook with 120 left-hand pages and 120 right-hand pages. Iris takes 4 minutes to read each page, while Ivy takes 5 minutes to read a left-hand page and 3 minutes to read a right-hand page.

   Iris and Ivy move onto the next page only when both sisters have completed reading. If a sister finishes reading a page first, the other sister will start reading three times as fast until she completes the page.

   How many minutes after they start reading will both sisters finish the textbook?

6. Let $\triangle ABC$ be an equilateral triangle with side length 24. Then, let M be the midpoint of BC. Define $\mathcal{P}$ to be the set of all points P such that $2PM = BC$. The minimum value of AP can be expressed as $\sqrt{a} - b$, where a and b are positive integers. Find $a + b$.

7. Jonathan has 10 songs in his playlist: 4 rap songs and 6 pop songs. He will select three unique songs to listen to while he studies. Let p be the probability that at least two songs are rap, and let q be the probability that none of them are rap. Find $\frac{p}{q}$.

8. A number K is called *6,8-similar* if K written in base 6 and K written in base 8 have the same number of digits. Find the number of 6,8-similar values between 1 and 1000, inclusive.

9. Quadrilateral $ABCD$ has $\angle ABC = 90^\circ$, $\angle ADC = 120^\circ$, $AB = 5$, $BC = 18$, and $CD = 3$. Find AD^2.

10. Bob, Eric, and Raymond are playing a game. Each player rolls a fair 6-sided die, and whoever has the highest roll wins. If players are tied for the highest roll, the ones that are tied reroll until one wins. At the start, Bob rolls a 4. The probability that Eric wins the game can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

11. Define the following infinite sequence s:

$$s = \left\{ \frac{9}{2}, \frac{99}{2^2}, \frac{999}{2^3}, \cdots, \frac{\overbrace{999...999}^{k \text{ nines}}}{2^k}, \cdots \right\}$$

   The sum of the first 2024 terms in s, denoted S, can be expressed as

$$S = \frac{5^a - b}{4} + \frac{1}{2^c},$$

   where a, b, and c are positive integers. Find $a + b + c$.

12. Andy is adding numbers in base 5. However, he accidentally forgets to write the units digit of each number. If he writes all the consecutive integers starting at 0 and ending at 50 (base 10) and adds them together, what is the difference between Andy's sum and the correct sum? (Express your answer in base-10.)

13. Let n be the positive real number such that the system of equations

$$y = \frac{1}{\sqrt{2024 - x^2}}$$
$$y = \sqrt{x^2 - n}$$

has exactly two real solutions for (x, y): (a, b) and $(-a, b)$. Then, $|a|$ can be expressed as $j\sqrt{k}$, where j and k are integers such that k is not divisible by any perfect square other than 1. Find $j \cdot k$.

14. Nakio is playing a game with three fair 4-sided dice. But being the cheater he is, he has secretly replaced one of the three die with his own 4-sided die, such that there is a 1/2 chance of rolling a 4, and a 1/6 chance to roll each number from 1 to 3. To play, a random die is chosen with equal probability and rolled. If Nakio guesses the number that is on the die, he wins.

    Unfortunately for him, Nakio's friends have an anti-cheating mechanism in place: when the die is picked, they will roll it three times. If each roll lands on the same number, that die is thrown out and one of the two unused dice is chosen instead with equal probability.

    If Nakio always guesses 4, the probability that he wins the game can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Find $m + n$.

15. A particle starts in the center of a 2m-by-2m square. It moves in a random direction such that the angle between its direction and a side of the square is a multiple of 30°. It travels in that direction at 1m/s, bouncing off of the walls of the square. After a minute, the position of the particle is recorded. The expected distance from this point to the start point can be written as

$$\frac{1}{a}\left(b - c\sqrt{d}\right),$$

where a and b are relatively prime, and d is not divisible by any perfect square. Find $a + b + c + d$.