Acton-Boxborough Math Competition 2022 Solutions

ABMC Team

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Speed Round

1. **Problem:** Alisha has 6 cupcakes and Tyrone has 10 brownies. Tyrone gives some of his brownies to Alisha so that she has three times as many desserts as Tyrone. How many desserts did Tyrone give to Alisha?

   Solution: Let x be the number of brownies Tyrone gives to Alisha. Since Alisha then has 3 times as many desserts, we can set up the equation

   $$3(10 - x) = (6 + x).$$

   Thus, $30 - 3x = 6 + x \Rightarrow\ 24 = 4x \Rightarrow\ x = \boxed{6}$.

   Proposed by Anusha Senapati

2. **Problem:** Bisky adds one to her favorite number. She then divides the result by 2, and gets 56. What is her favorite number?

   Solution: Let x be her favorite number. We get $\dfrac{x+1}{2} = 56 \Rightarrow\ x = \boxed{111}$.

   Proposed by Anusha Senapati

3. **Problem:** What is the maximum number of points at which a circle and a square can intersect?

   Solution: A circle can be drawn to go through two points on each side of the square, so the answer is $\boxed{8}$.

   Proposed by Jerry Tan

4. **Problem:** An integer N leaves a remainder of 66 when divided by 120. Find the remainder when N is divided by 24.

   Solution: If N leaves a remainder of 66 when divided by 120, then N can be written has $66 + 120k$ for an integer k. Then, $66 + 120k$ can be rewritten as $18 + 24 + 24 + 24 \cdot 5k = 18 + 24(2 + 5k)$, so N leaves a remainder of $\boxed{18}$ when divided by 24.

   Proposed by Jerry Li

5. **Problem:** 7 people are chosen to run for student council. How many ways are there to pick 1 president, 1 vice president, and 1 secretary?

   Solution: The number of ways this can occur is $7 \cdot 6 \cdot 5 = \boxed{210}$.

   Proposed by Anusha Senapati

6. **Problem:** Anya, Beth, Chloe, and Dmitri are all close friends, and like to make group chats to talk. How many group chats can be made if Dmitri, the gossip, must always be in the group chat and Anya is never included in them? Group chats must have more than one person.

   Solution: We can ignore Dmitri and Anya because their presence in the chats are invariable. For the other two, the options are:

   Only Chloe
   Only Beth
   Both Beth and Chloe

   Thus, $\boxed{3}$.

   Proposed by Anusha Senapati

7. **Problem:** There exists a telephone pole of height 24 feet. From the top of this pole, there are two wires reaching the ground in opposite directions, with one wire 25 feet, and the other wire 40 feet. What is the distance (in feet) between the places where the wires hit the ground?

   Solution: In order to solve this problem, it should be noted that the wires reaching the ground create two right triangles. The first right triangle, which consists of a 25 feet long hypotenuse leads us to use the Pythagorean theorem in order to solve. From here, if the leg is denoted as x, for example, then the equation we get is: $(25)^2 = (x)^2 + (24)^2$. Solving this, we get $x = 7$. Similarly, we can apply the same method of solving for the 40 foot wire and get the equation:$(40)^2 = (x)^2 + (24)^2$. Using the 24-32-40 Pythagorean triple, the answer appears to be $x = 32$. Finally, we are asked to find the distance between both of these wires. So, the sum of the legs must be added in order to find this distance and the answer is found: 32 + 7 = $\boxed{39}$ feet.

   Proposed by Ryon Das

8. **Problem:** Tarik is dressing up for a job-interview. He can wear a chill, business, or casual outfit. If he wears a chill oufit, he must wear a t-shirt, shorts, and flip-flops. He has eight of the first, seven of the second, and three of the third. If he wears a business outfit, he must wear a blazer, a tie, and khakis; he has two of the first, six of the second, and five of the third; finally, he can also choose the casual style, for which he has three hoodies, nine jeans, and two pairs of sneakers. How many different combinations are there for his interview?

   Solution: We need to use casework to solve this problem. Case 1 is chill outfit. In this case, there are 168 combinations, a number we can derive by multiplying each number of the article of clothing. A similar approach follows for the other two cases, and we see that business yields 60, and that casual has 54 probabilities. $168 + 60 + 54 = \boxed{282}$ different combinations for his interview.

   Proposed by Ryon Das

9. **Problem:** If a non-degenerate triangle has sides 11 and 13, what is the sum of all possibilities for the third side length, given that the third side has integral length?

   Solution: The third side must satisfy the following equations by the triangle inequality:

$$x + 11 > 13$$
$$x + 13 > 11$$
$$11 + 13 > x.$$

   Only the first and last inequality matters; we find $2 < x < 24$, so the integral values x can take are from 3 to 23. The sum of those values is $21 \cdot (26/2) = \boxed{273}$.

   Proposed by Anusha Senapati

10. **Problem:** An unknown disease is spreading fast. For every person who has the this illness, it is spread on to 3 new people each day. If Mary is the only person with this illness at the start of Monday, how many people will have contracted the illness at the end of Thursday?

    Solution: For each person who gets the disease, it spreads to 3 people, so each day, we multiply the number of people who has it by 4 (not 3 because the person who is spreading it still has it so):

    End of Monday: 4
    End of Tuesday: 16
    End of Wednesday: 64
    End of Thursday: $\boxed{256}$

    Proposed by Lakshika Kamalaganesh

11. **Problem:** Gob the giant takes a walk around the equator on Mars. If Gob's head is $\frac{13}{\pi}$ meters above his feet, how much farther (in meters) does his head travel than his feet?

    Solution: Let r be the radius of mars. Then, Gob's head travels the circumference of a circle with radius $r + \frac{13}{\pi}$, which is $2\pi\left(r + \frac{13}{\pi}\right) = 2\pi r + 26$ meters. Gob's feet travels the circumference of a circle with radius r, or $2\pi r$ meters. Hence, his head travels $2\pi r + 26 - 2\pi r = \boxed{26}$ more meters than his feet.

    Proposed by Jeremy Wen

12. **Problem:** 2022 leaves a remainder of $2, 6, 9,$ and 7 when divided by $4, 7, 11,$ and 13 respectively. What is the next positive integer which has the same remainders to these divisors?

    Solution: The LCM of $4, 7, 11, 13$ is 4004. If we add this number to 2022, we get $\boxed{6026}$, the number which has the same remainders.

    Proposed by Ryon Das

13. **Problem:** In triangle ABC, $AB = 20, BC = 21$, and $AC = 29$. Let D be a point on AC such that $\angle ABD = 45°$. If the length of AD can be represented as $\frac{a}{b}$, what is $a + b$?

    Solution: The three sides are a Pythagorean triple, and thus form a right triangle with right angle at B (opposite to longest side $\overline{AC}$).$\angle ABC$ is bisected by $\overline{AD}$, as $\angle ABD$ is 45°, which is half of 90°. The angle bisector theorem tells us $\overline{AD}$ splits $\overline{AC}$ into the ratio of the sides, or 20:21. $29\frac{20}{20+21} = \frac{580}{41}$.$580 + 41 = \boxed{621}$

    Proposed by Ryon Das

14. **Problem:** Find the number of primes less than 100 such that when 1 is added to the prime, the resulting number has 3 divisors.

    Solution: An positive integer has that has 3 divisors is a perfect square. Thus, p must satisfy $p + 1 = n^2$ for a positive integer n. This simplifies to $p = n^2 - 1 = (n + 1)(n - 1)$. Note that for p to be a prime, then one of $n + 1, n - 1$ must be ± 1. If $n + 1 = 1$, then we obtain $n = 0$ and $p = 1$ but 1 is not a prime. If $n - 1 = 1$ then we obtain $n = 2$, which yields $p = 3$. Hence, there is only $\boxed{1}$ prime that satisfies the problem conditions.

    Proposed by Jerry Li

15. **Problem:** What is the coefficient of the term a^4z^3 in the expanded form of $(z - 2a)^7$?

    Solution: Using the binomial theorem, the term with z^3 in the expanded form is $\binom{7}{3}z^3(-2a)^4 = 35 \cdot z^3 \cdot 16a^4 = 560a^4z^3$. The coefficient is therefore $\boxed{560}$.

    Proposed by Anusha Senapati

16. **Problem:** Let ℓ and m be lines with slopes $-2, 1$ respectively. Compute $|s_1 \cdot s_2|$ if s_1, s_2 represent the slopes of the two distinct angle bisectors of ℓ and m.

    Solution: Notice that the two angle bisectors of the lines l and m must be perpendicular. This means that their slopes are negative reciprocals of each other and therefore multiply to -1. So $s_1 \cdot s_2 = -1$ and $|s_1 \cdot s_2| = \boxed{1}$.

    Proposed by Jerry Li

17. **Problem:** R1D2, Lord Byron, and Ryon are creatures from various planets. They are collecting monkeys for King Avanish, who only understands octal (base 8). R1D2 only understands binary (base 2), Lord Byron only understands quarternary (base 4), and Ryon only understands decimal (base 10).

R1D2 says he has 101010101 monkeys and adds his monkey to the pile. Lord Byron says he has 3231 monkeys and adds them to the pile. Ryon says he has 576 monkeys and adds them to the pile. If King Avanish says he has x monkeys, what is the value of x?

Solution: Let's first convert all of the numbers to decimal so we can add them more easily:

$$101010101_2 = 2^8 + 2^6 + 2^4 + 2^2 + 2^0 = 341$$
$$3231_4 = 3 \cdot 4^3 + 2 \cdot 4^2 + 3 \cdot 4^1 + 1 \cdot 4^0 = 237$$
$$576_{10} = 576.$$

Adding them together, we get $341 + 237 + 576 = 1154$. Now we have to convert this to base 8 since King Avanish only understands that. The highest power of 8 that fits into it is $8^3 = 512$, and two 512s fit into 1154. If we repeatedly continue this process, we end up writing 1154 as $2 \cdot 8^3 + 2 \cdot 8^2 + 2 \cdot 8^0$, giving us a base 8 representation of $\boxed{2202}$.

Proposed by Avanish Gowrishankar

18. **Problem:** A quadrilateral is defined by the origin, $(3,0)$, $(0,10)$, and the vertex of the graph of $y = x^2 - 8x + 22$. What is the area of this quadrilateral?

    Solution: The x-coordinate of the vertex of $x^2 - 8x + 22$ is $\frac{-(-8)}{2(1)} = 4$ (since the x-coordinate of the vertex is $\frac{-b}{2a}$). Plugging 4 back in, we get that the vertex has coordinates $(4,6)$. To find the area of the quadrilateral, we split the quadrilateral along the diagonal formed by the origin and $(4,6)$. The area of the triangle containing the point $(0,10)$ is $10 \cdot 4/2 = 20$, and the area of the triangle containing the point $(3,0)$ is $3 \cdot 6/2 = 9$. Adding these areas together, we get $20 + 9 = \boxed{29}$.

    Proposed by Ryon Das

19. **Problem:** There is a sphere-container, filled to the brim with fruit punch, of diameter 6. The contents of this container are poured into a rectangular prism container, again filled to the brim, of dimensions 2π by 4 by 3. However, there is an excess amount in the original container. If all the excess drink is poured into conical containers with diameter 4 and height 3, how many containers will be used?

    Solution: The volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 3^3 = 36\pi$. The volume of the rectangular container is $2\pi \cdot 4 \cdot 3 = 24\pi$. So the excess volume is $36\pi - 24\pi = 12\pi$. The volume of each of the cones is $\pi r^2 \cdot h/3 = \pi 2^2 \cdot 3/3 = 4\pi$. Finally, this means that the excess will fill $12\pi/4\pi = \boxed{3}$ containers.

    Proposed by Ryon Das

20. **Problem:** Brian is shooting arrows at a target, made of concurrent circles of radius $1, 2, 3$, and 4. He gets 10 points for hitting the innermost circle, 8 for hitting between the smallest and second smallest circles, 5 for between the second and third smallest circles, 2 points for between the third smallest and outermost circle, and no points for missing the target. Assume for each shot he takes, there is a 20% chance Brian will miss the target, but otherwise the chances of hitting each target are proportional to the area of the region. The chance that after three shots, Brian will have scored 15 points can be expressed as $\dfrac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

    Solution: Let's first calculate the probability of hitting each ring. The area of the innermost ring is $1^2\pi = \pi$. The area of the second ring is $2^2\pi - 1^2\pi = 3\pi$, third ring is $3^2\pi - 2^2\pi = 5\pi$, and the outermost ring is $4^2\pi - 3^2\pi = 7\pi$. The probability of hitting the innermost ring is therefore $\frac{4}{5} \cdot \frac{\pi}{16\pi} = \frac{1}{20}$ (the probability of not missing times the ratio of the circle area to total area). By a similar process, we end up getting the following probabilities for scoring each amount of points:

    0 points: $\frac{1}{5}$
    2 points: $\frac{7}{20}$
    5 points: $\frac{1}{4}$

8 points: $\frac{3}{20}$
10 points: $\frac{1}{20}$

Moving on, there are 3 cases on how we can score 15 points in three tries.

Case 1: We score $0, 5, 10$ points in some order. The probability of this happening is $\frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{20} \cdot 3! = \frac{3}{200}$. Note that we multiplied by 3! because we can score these 3 points in any order.

Case 2: We score $2, 5, 8$ points in some order. The probability of this happening is $\frac{7}{20} \cdot \frac{1}{4} \cdot \frac{3}{20} \cdot 3! = \frac{63}{800}$.

Case 3: We score $5, 5, 5$ points. The probability of this happening is $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$. Notice that we don't multiply by 3! here because all of the points are the same, so there only is one order to score them.

Adding the probabilities together, we get $\frac{3}{200} + \frac{63}{800} + \frac{1}{64} = \frac{7}{64}$. Thus, the answer is $7 + 64 = \boxed{71}$.

Proposed by Ryon Das

21. **Problem:** What is the largest integer value of n such that $\frac{2n^3+n^2+7n-15}{2n+1}$ is an integer?

Solution: We see that $\dfrac{2n^3 + n^2 + 7n - 15}{2n+1} = n^2 + \dfrac{7n-15}{2n+1} = n^2 + 3 + \dfrac{n-18}{2n+1}$, so we need to find the largest integer n such that $\dfrac{n-18}{2n+1}$ is an integer. If $2n+1$ is a divisor of $n-18$, then $|n-18| \geq |2n+1|$ or $n - 18 = 0$.

Case 1: Let us consider when $|n-18| \geq |2n+1|$. If $n \geq 18$ then $n - 18 \geq 2n+1 \Rightarrow n \leq -19$ which obviously yields no values of n. Otherwise, if $-\dfrac{1}{2} \leq n < 18$ then we obtain the inequality $18 - n \geq 2n + 1 \Rightarrow n \leq \dfrac{17}{3} = 5\dfrac{2}{3}$. Combining $-\dfrac{1}{2} \leq n < 18$ with $n \leq 5\dfrac{2}{3}$ we obtain $-\dfrac{1}{2} \leq n \leq 5\dfrac{2}{3}$. Then, we test values of n from 5 to 0 to find a value of n that may work for the largest possible value of n. We find that $n = 5$ yields $\dfrac{-13}{11}$, $n = 4$ yields $\dfrac{-14}{9}$, $n = 3$ yields $\dfrac{-15}{7}$, $n = 2$ yields $\dfrac{-16}{5}$, $n = 1$ yields $\dfrac{-17}{3}$, $n = 0$ yields $\dfrac{-18}{1}$. The largest value of n such that it is an integer $n = 0$ for this case.

Case 2: This case just yields $n = 18$.

Since $18 > 0$, $\boxed{18}$ is the largest integer value of n that yields an integer.

Proposed by Alice Hui

22. **Problem:** Let $f(x, y) = x^3 + x^2y + xy^2 + y^3$. Compute

$$f(0,2) + f(1,3) + \cdots + f(9,11)$$

Solution: We can rewrite $f(x,y) = \dfrac{y^4 - x^4}{y - x}$. Then, we see that $f(0,2) + f(1,3) + \cdots f(9,11) = \dfrac{2^4 - 0^4}{2} + \dfrac{3^4 - 1^4}{2} + \cdots + \dfrac{10^4 - 8^4}{2} + \dfrac{11^4 - 9^4}{2}$ in which most of the terms telescope (cancel out), leaving us with $\dfrac{11^4 + 10^4 - 1^4 - 0^4}{2} = \boxed{12320}$.

Proposed by Jerry Li

23. **Problem:** Let $\triangle ABC$ be a triangle. Let AM be a median from A. Let the perpendicular bisector of segment $\overline{AM}$ meet AB and AC at D, E respectively. Given that $AE = 7, ME = MC$, and $BDEC$ is cyclic, then compute AM^2.

Solution: Since M is the midpoint of BC, we have $MB = MC$. We are given that $ME = MC$, so $MB = MC = ME$, which implies M is the center of cyclic quadrilateral $BCDE$. Hence, $ME = MD$ as well. Since DE is the perpendicular bisector of AM, we know $AE = EM = MD = DA$. This means that $\angle BAM = \angle CAM$. Combined with the fact that M is the midpoint of BC, we can deduce $\triangle ABC$ is isosceles, with $AB = AC$. Since $\triangle AMC$ is a right triangle, we have $AD = EC = 7 \Rightarrow AC = 14$. Furthermore, $MC = 7$. Thus, $AM^2 = 14^2 - 7^2 = \boxed{147}$.

Remark: The condition $ME = MC$ is extra information given. Even without this information, the reader can deduce through trigonometry that $\triangle ABC$ must either be isosceles, or right at $\angle A$.

Proposed by Jerry Li

24. **Problem:** Compute the number of ordered triples of positive integers (a, b, c) such that $a \leq 10, b \leq 11, c \leq 12$ and $a > b - 1$ and $b > c - 1$

    Solution:

    There is a bijection between number of ordered triples of positive integers (a, b, c) such that $a \leq 10, b \leq 11, c \leq 12$ and $a > b - 1$ and $b > c - 1$ (let elements of this type belong to set A) and the number of ordered triples of positive integers (a, b, c) such that a, b, c are in descending order, and are all less than or equal to 12 (let elements of this type belong to set B).

    To show this, we need to show each ordered triple in A corresponds to an ordered triple in set B, and each ordered triple in set B corresponds to an ordered triple in set A. Let (a, b, c) be an ordered triple in set A. Then, $(a + 2, b + 1, c)$ must be an ordered triple in set B, since $a > b - 1 \Rightarrow a + 2 > b + 1$ and $b > c - 1 \Rightarrow b + 1 > c$, so $a + 2 > b + 1 > c$. Furthermore, $a + 2 \leq 12, b + 1 \leq 12, c \leq 12$, matching the second condition to be in B. Going the other way, let (a, b, c) be an ordered triple in set B. Then, $(a - 2, b - 1, c)$ must be an ordered triple in set A, since $a > b \Rightarrow a - 2 > (b - 1) - 1$ and $b > c \Rightarrow b - 1 > c - 1$. Additionally, $a \leq 12, b \leq 12, c \leq 12 \Rightarrow a - 2 \leq 10, b - 1 \leq 11, c \leq 10$, so $(a - 2, b - 1, c)$ indeed is an element of set A.

    We have shown that the number of ordered triples in set A is equal to the number of ordered triples in set B. The number of triples in set B, however, is simply $\binom{12}{3} = \boxed{220}$, since we choose any 3 distinct numbers less than or equal to 12, and then we order the numbers in descending order to obtain our triple (a, b, c) in B.

    Proposed by Jerry Li

25. **Problem:** For a positive integer n, denote by $\sigma(n)$ the the sum of the positive integer divisors of n. Given that $n + \sigma(n)$ is odd, how many possible values of n are there from 1 to 2022, inclusive?

    Solution: Let $\sigma(x)$ denote the sum of the divisors of a positive integer x.

    n has some number of even divisors and odd divisors. Note that the even divisors do not affect the parity of $\sigma(n)$, while the odd divisors toggle the parity from odd to even or even to odd. This means $\sigma(n)$ is odd when the number of odd divisors is odd, and otherwise it is even.

    We can write $n = 2^i \cdot j$, where i is the largest integer value that makes j an integer. The number of odd divisors of n is just the number of divisors of j. The number of divisors of j is odd when it is a square, and it is even otherwise. So $\sigma(n)$ is odd if and only if j is a square.

    For $n + \sigma(n)$ to be odd, either n is even and $\sigma(n)$ is odd, or vice versa. In the first case, n is even, so $i > 0$, and $\sigma(n)$ is odd, so j is a square. In the second case, n is odd, so $i = 0$, and $\sigma(n)$ is even, so j is not a square. So we want n that are a power of 2 (that is not 1) times an odd square, or n that are odd and not squares.

    In the first case, we can use casework. For each power of 2, we find an upper bound on the odd squares by dividing 2022 by the power of 2. The results are summarized in the table below.

Power of 2	Odd Squares	Total
2	$1^2, 3^2, \ldots, 31^2$	16
4	$1^2, 3^2, \ldots, 21^2$	11
8	$1^2, 3^2, \ldots, 15^2$	8
16	$1^2, 3^2, \ldots, 11^2$	6
32	$1^2, 3^2, 5^2, 7^2$	4
64	$1^2, 3^2, 5^2$	3
128	$1^2, 3^2$	2
256	1^2	1
512	1^2	1
1024	1^2	1

In total, this case has $16 + 11 + 8 + 6 + 4 + 3 + 2 + 1 + 1 + 1 = 53$ numbers.

In the second case, we can take all of the odd numbers less than 2022 and remove those that are squares. There are 1011 odd numbers less than 2022. 43^2 is the largest odd square less than 2022, so there are 22 odd squares less than 2022. Then this case has $1011 - 22 = 989$ numbers.

In total, there are $53 + 989 = \boxed{1042}$ possibilities for n.

Proposed by Matthew Qian

Accuracy Round

1. **Problem:** Let $X = 2022 + 022 + 22 + 2$. When X is divided by 22, there is a remainder of R. What is the value of R?

   Solution: The expression sums to 2068. To quickly check if a number is divisible by 22, we can combine the divisibility rules of 11 and 2. Because the first and third digits sum to the same as the second and fourth digits in 2068, it must be divisible by 11. And because 2068 is even, it is divisible by 2. Therefore, it must be divisible by 22, which means that there is a remainder of $\boxed{0}$.

   Proposed by Ryon Das

2. **Problem:** When Amy makes paper airplanes, her airplanes fly 75% of the time. If her airplane flies, there is a $\frac{5}{6}$ chance that it won't fly straight. Given that she makes 80 airplanes, what is the expected number airplanes that will fly straight?

   Solution: Of the 80 made, 75 percent, or 60 airplanes, will fly . Of the airplanes that fly, only a sixth will fly straight. This means that $\boxed{10}$ airplanes will fly straight.

   Proposed by Lakshika Kamalaganesh

3. **Problem:** It takes Joshua working alone 24 minutes to build a birdhouse, and his son working alone takes 16 minutes to build one. The effective rate at which they work together is the sum of their individual working rates. How long in seconds will it take them to make one birdhouse together?

   Solution: For these types of problems, we need to find a common interval and go from there. Let's take one minute. In that time, Joshua can make $\frac{1}{24}$ of a birdhouse, and his son can make $\frac{1}{16}$ of one. Adding these together, we see that they can together make $\frac{5}{48}$ of a birdhouse in a minute. $5 \cdot \frac{48}{5} = 48$, so we know that it will take $\frac{48}{5}$ minutes. This times 60 seconds per minute means that together, it will take them $\boxed{576}$ seconds to make one birdhouse.

   Proposed by Anusha Senapati

4. **Problem:** If Katherine's school is located exactly 5 miles southwest of her house, and her soccer tournament is located exactly 12 miles northwest of her house, how long, in hours, will it take Katherine to bike to her tournament right after school given she bikes at 0.5 miles per hour? Assume she takes the shortest path possible.

   Solution: A well-known Pythagorean triple is of the lengths $5 - 12 - 13$. Since we see this here, we know that the shortest path is 13 miles. Since Katherine travels at the very fast speed of 0.5 miles per hour, we know that it will take her $\frac{13}{0.5} = \boxed{26}$ hours to travel to her tournament.

   Proposed by Anusha Senapati

5. **Problem:** What is the largest possible integer value of n such that $\frac{4n+2022}{n+1}$ is an integer?

   Solution: In order for this fraction to be equivalent to an integer, $4n + 2022$ has to be divisible by $n + 1$. We know that $4n + 4$ is divisible by $n + 1$. Therefore, we can subtract $4n + 4$ from $4n + 2022$ and conclude that it is sufficient for 2018 to be divisible by $n + 1$, and the largest integer that $n + 1$ can be is 2018 in order for that statement to be true. Thus, the largest integer that n can be is $\boxed{2017}$

   Proposed by Alice Hui

6. **Problem:** A caterpillar wants to go from the park, situated at $(8, 5)$, to his home, located at $(4, 10)$. He wants to avoid routes through $(6, 7)$ and $(7, 10)$. How many possible routes are there if the caterpillar can move in the north and west directions, one unit at a time?

   Solution: The total number of paths from $(8, 5)$ to $(4, 10)$ without worrying about points we can't go through is $\binom{9}{4} = 126$ (because we need to move west 4 times and north 5 times, so need to choose 4 west moves out of a total of 9 moves). Now we need to subtract paths that do go through forbidden points. Notice that no path can go through both forbidden points $(6,7)$ and $(7,10)$, since it would require moves in the south or east direction. This allows us to count the paths going through them separately.

The paths that go through $(6,7)$ is the number of paths from $(8,5)$ to $(6,7)$ times the number of paths from $(6,7)$ to $(4,10)$. This is $\binom{4}{2}\binom{5}{2} = 60$.

The paths that go through $(7,10)$ is the number of paths from $(8,5)$ to $(7,10)$ times the number of paths from $(7,10)$ to $(4,10)$. This is $\binom{6}{1}\binom{3}{0} = 6$.

So the number paths that go through forbidden points are $60 + 6 = 66$, meaning that the number of paths that don't go through those points is $126 - 66 = \boxed{60}$.

Proposed by Ryon Das

7. **Problem:** Let $\triangle ABC$ be a triangle with $AB = 2\sqrt{13}, BC = 6\sqrt{2}$. Construct square $BCDE$ such that $\triangle ABC$ is not contained in square $BCDE$. Given that $ACDB$ is a trapezoid with parallel bases $\overline{AC}, \overline{BD}$, find AC.

   Solution: Since $BCDE$ is a square, $\angle DBC = 45°$. Then, because $\overline{AC} \parallel \overline{BD}, \angle ACB = \angle DBC = 45°$. Now, let's drop an altitude from B to $\overline{AC}$ at point F. Notice that $\triangle BCF$ is a $45 - 45 - 90$ triangle, meaning $CF = BF = \frac{6\sqrt{2}}{\sqrt{2}} = 6$. Then using Pythagorean Theorem on $\triangle ABF$, $AF = \sqrt{AB^2 - BF^2} = \sqrt{(2\sqrt{13})^2 - 6^2} = 4$. As a result, $AC = AF + FC = 4 + 6 = \boxed{10}$.

   Proposed by Jerry Li

8. **Problem:** How many integers a with $1 \le a \le 1000$ satisfy $2^a \equiv 1 \pmod{25}$ and $3^a \equiv 1 \pmod{29}$?

   Solution: Let us observe that the least integer a where $2^a \equiv 1 \pmod{25}$ is 20 and the least integer a where $3^a \equiv 1 \pmod{29}$ is 28. Therefore we can conclude that in order to satisfy both statements, a must be a multiple of both 20 and 28. We see that the LCM of 28 and 20 is 140, so that a must be a multiple of 140. There are $\boxed{7}$ multiples of 140 which are less than or equal to 1000

   Proposed by Jerry Li

9. **Problem:** Let $\triangle ABC$ be a right triangle with right angle at B and $AB < BC$. Construct rectangle $ADEC$ such that $\overline{AC}, \overline{DE}$ are opposite sides of the rectangle, and B lies on $\overline{DE}$. Let $\overline{DC}$ intersect $\overline{AB}$ at M and let $\overline{AE}$ intersect $\overline{BC}$ at N. Given $CN = 6, BN = 4$, find the $m + n$ if MN^2 can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n.

   Solution: Let $AC = 3x$. Since $\triangle CAN \sim \triangle BEN \Rightarrow \dfrac{CA}{BE} = \dfrac{CN}{BN} = \dfrac{6}{4} = \dfrac{3}{2} \Rightarrow BE = 2x$. Furthermore, since $\triangle CBE \sim \triangle ACB$, we have $\dfrac{CB}{AC} = \dfrac{BE}{CB} \Rightarrow \dfrac{10}{3x} = \dfrac{2x}{10}$. We find $x = \sqrt{\dfrac{100}{6}} = \dfrac{10}{\sqrt{6}}$. Thus, $AC = 3x = \dfrac{30}{\sqrt{6}} = 5\sqrt{6}$. Applying Pythagorean theorem on $\triangle ABC$ yields $AB^2 + BC^2 = AC^2 = AB^2 + 100 = 150 \Rightarrow AB = 5\sqrt{2}$. Furthermore, $DB = AC - BE = 3x - 2x = x$. We also have $\triangle AMC \sim \triangle BMD \Rightarrow \dfrac{AC}{BD} = \dfrac{AM}{BM} = 3$. Note that $AM + BM = 5\sqrt{2}$, so $AM = 5\sqrt{2} \cdot \dfrac{3}{4}, BM = 5\sqrt{2} \cdot \dfrac{1}{4}$. To find MN^2, we use Pythagorean theorem on $\triangle BMN$, to obtain $MN^2 = 3^2 + \left(\dfrac{5\sqrt{2}}{4}\right)^2 = 9 + \dfrac{25}{8} = \dfrac{97}{8}$. Thus, $m + n = \boxed{105}$.

   Proposed by Jerry Li

10. **Problem:** An elimination-style rock-paper-scissors tournament occurs with 16 players. The 16 players are all ranked from 1 to 16 based on their rock-paper-scissor abilities where 1 is the best and 16 is the worst. When a higher ranked player and a lower ranked player play a round, the higher ranked player always beats the lower ranked player and moves on to the next round of the tournament. If the initial order of players are arranged randomly, and the expected value of the rank of the 2nd place player of the tournament can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n what is the value of $m + n$?

    Solution: The tournament can be divided into 2 sides of 8: one with player 1 and one without. Note that the best player in the side without player 1 will always win second place. If it's player 2, then there are $\binom{14}{7}$ ways to set up the rest of the bracket.

    If it's player 3, player 2 must be in the side with player 1. So, there are $\binom{13}{7}$ ways to set it up. Continuing, we get that for player 4, there are $\binom{12}{7}$ ways, and this continues until player 9 where there are $\binom{7}{7}$ ways. Overall, there are $\frac{\binom{16}{8}}{2}$ setups. So, the expected value is:

    $$\frac{2\binom{14}{7} + 3\binom{13}{7} + \cdots + 9\binom{7}{7}}{\frac{\binom{16}{8}}{2}}.$$

    Using Hockey Stick Identity to simplify we get:

    $$\frac{\binom{15}{8} + \binom{15}{8} + \binom{14}{8} + \cdots \binom{8}{8}}{\frac{\binom{16}{8}}{2}} = \frac{\binom{15}{8} + \binom{16}{9}}{\frac{\binom{16}{8}}{2}} = \frac{17875}{6435} = \frac{25}{9} \to \boxed{34}.$$

    Proposed by Lakshika Kamalaganesh

11. **Problem:** **Estimation:** Estimate the number of twin primes (pairs of primes that differ by 2) where both primes in the pair are less than 220022.

    Solution: Answer is $\boxed{372}$. Here is the code I used:

```
    #include <iostream>
using namespace std;

bool isPrime(int n) //taken directly from geeksforgeeks
{

    if (n <= 1)
        return false;

    // Check from 2 to n-1
    for (int i = 2; i < n; i++)
        if (n % i == 0)
            return false;
```

```
    return true;
}

int main(){
  int count = 0;
  for(int i = 1; i < 22022; i++){
    if(isPrime(i) && isPrime(i+2)){
      count++;
    }
  }
  cout << count;
}
```

Proposed by Jerry Li

Team Round

Round 1

1. **Problem:** If the sum of two non-zero integers is 28, then find the largest possible ratio of these numbers.

   Solution: If we select 27 and 1, we see they form the largest possible ratio of $\frac{27}{1}$, or $\boxed{27}$.

   Proposed by Anusha Senapati

2. **Problem:** If Tom rolls a eight-sided die where the numbers $1-8$ are all on a side, let $\dfrac{m}{n}$ be the probability that the number is a factor of 16 where m, n are relatively prime positive integers. Find $m+n$.

   Solution: The factors of 16 less than equal to 8 are $1, 2, 4$, and 8. There are four appropriate values, and eight possibilities, so there is a $\frac{4}{8}$, or $\frac{1}{2}$ chance that a factor of 16 is rolled. $m = 1$ and $n = 2$, so $m + n = \boxed{3}$.

   Proposed by Anusha Senapati

3. **Problem:** The average score of 35 second graders on an IQ test was 180 while the average score of 70 adults was 90. What was the total average IQ score of the adults and kids combined?

   Solution: Since the average of a group can be defined as the sum of all the values divided by the number of values, we know that the scores of the second-graders add up to $35 \cdot 180 = 6300$. Similarly, the sums of the scores from the pool of adults amounts to $70 \cdot 90 = 6300$. Together, this sums to 12600, with 105 people contributing to this score. Dividing, we get that the average score is $\frac{12600}{105} = \boxed{120}$.

   Proposed by Anusha Senapati

Round 2

1. **Problem:** So far this year, Bob has gotten a 95 and a 98 in Term 1 and Term 2. How many different pairs of Term 3 and Term 4 grades can Bob get such that he finishes with an average of 97 for the whole year? Bob can only get integer grades between 0 and 100, inclusive.

   Solution: To get a 97, he must get have totaled 388. He has already scored 193, which means that in the last two tests he needs to score a combined 195. There are six different possibilities: him scoring $95, 96, 97, 98, 99$, and 100 on the Term 3 grade. This means that there are $\boxed{6}$ different pairs.

   Proposed by Ryon Das

2. **Problem:** If a complement of an angle M is one-third the measure of its supplement, then what would be the measure (in degrees) of the third angle of an isosceles triangle in which two of its angles were equal to the measure of angle M?

   Solution: If the complement of $\angle M$ is x°, $x + 3x = 180^\circ$, so $x = 45$ degrees. This means $\angle M$ is $90^\circ - 45^\circ$, or 45°. $\frac{180-2\cdot 45}{2} = \boxed{90}$ degrees.

   Proposed by Anusha Senapati

3. **Problem:** The distinct symbols $\heartsuit$, $\diamondsuit$, $\clubsuit$ and $\spadesuit$ each correlate to one of $+$, $-$, $\times$, $\div$, not necessarily in that given order. Given that

   $$((((72 \diamondsuit 36) \spadesuit 0\ ) \diamondsuit 32) \clubsuit 3)\ \heartsuit 2 = 6,$$

   what is the value of

   $$(((((64 \spadesuit 8) \heartsuit 6) \clubsuit 5) \heartsuit 1) \clubsuit 7) \diamondsuit 1?$$

Solution: By casework of the given equation, we can find that $\diamondsuit \cong -$, $\spadesuit \cong +$, $\clubsuit \cong \times$, and $\heartsuit \cong \div$. Applying these operations to the second expression, we find it equivalent to $((\frac{(64+8)}{6} \times 5) \div 1) \times 7 - 1 = \boxed{419}$.

Proposed by Lakshika Kamalaganesh

Round 3

1. **Problem:** How many ways can 5 bunnies be chosen from 7 male bunnies and 9 female bunnies if a majority of female bunnies is required? All bunnies are distinct from each other.

   Solution: We can separate this into cases. Case 1 is that all five bunnies are female. We get 9 Choose 5 from this case, which is equal to 126. Case 2 is that four bunnies are female, and one bunny is a male. The female selection would yield 9 Choose 4, and the male selection would yield 7. Multiplying these two, we get $126 \cdot 7 = 882$. Case 4 is that three bunnies are female, and two are male. This case leads to 9 choose 3 for the girl bunnies and 7 choose 2 for the boy bunnies. $84 \cdot 21 = 1768$ possibilities from this case. Summing them all up, we see that $1764 + 882 + 126 = \boxed{2772}$.

   Proposed by Anusha Senapati

2. **Problem:** If the product of the LCM and GCD of two positive integers is 2021, what is the product of the two positive integers?

   Solution: The product of the LCM of two numbers and the GCD of those numbers is equivalent to the product of those two numbers. Thus, because the product of LCM and GCD of the two positive integers is 2021, the product of the postivie integers is $\boxed{2021}$.

   Proposed by Alice Hui

3. **Problem:** The month of April in ABMC-land is 50 days long. In this month, on 44% of the days it rained, and on 28% of the days it was sunny. On half of the days it was sunny, it rained as well. The rest of the days were cloudy. How many days were cloudy in April in ABMC-land?

   Solution: Applying the percentages, we see that it rained 22 days, and was sunny on 14 days. Half of the 14 sunny days is 7 days, which means there was a 7 day overlap. Thus, on 29 days it was sunny or rainy. This means that on the remaining $50 - 29 = \boxed{21}$ days it was cloudy.

   Proposed by Ryon Das

Round 4

1. **Problem:** In how many ways can 4 distinct dice be rolled such that a sum of 10 is produced?

   Solution: In order to solve this problem, we can list the possible results in which the 4 numbers sum to 10. From this we get the following distinct sequences:
   $2, 2, 2, 4$
   $2, 2, 3, 3$
   $1, 1, 2, 6$
   $1, 1, 3, 5$
   $1, 1, 4, 4$
   $1, 2, 2, 5$
   $1, 2, 3, 4$
   $1, 3, 3, 3$

   To find the possible combinations as to how these can be rolled, each sequence must be calculated separately to account for repeating numbers. For the first sequence, $2, 2, 2, 4$, the number of permutations can be calculated as $4!/3!$, since the digit 2 appears 3 times in the sequence. This results in a value of 4 possible permutations. Similarly, the remaining permutations can be calculated:

$\{2,2,3,3\} : 4!/(2 \cdot 2!) = 6$
$\{1,1,2,6\} : 4!/2! = 12$
$\{1,1,3,5\} : 4!/2! = 12$
$\{1,1,4,4\} : 4!/(2! \cdot 2!) = 6$
$\{1,2,2,5\} : 4!/2! = 12$
$\{1,2,3,4\} : 4! = 24$
$\{1,3,3,3\} : 4!/3! = 4$

To get the final answer, the amount of combinations can be totaled to receive an answer of $\boxed{80}$ total ways.

Proposed by Lakshika Kamalaganesh

2. **Problem:** If p, q, r are positive integers such that $p^3\sqrt{q}r^2 = 50$, find the sum of all possible values of pqr.

   Solution: We see 50, p, and r are all rational; therefore, $\sqrt{q}$ is rational. Thus, q is a perfect square, and $\sqrt{q}$ is an integer. p^3 must be a factor of 50, so $p = 1$. r^2 must be a factor of 50, so $r = 1$ or 5. If $r = 1$, then $\sqrt{q} = 50$, so $q = 2500$. If $r = 5$, then $\sqrt{q} = 2$, so $q = 4$. So $(1, 4, 5)$ and $(1, 2500, 1)$ are the only solutions. Our answer is $1 \cdot 4 \cdot 5 + 1 \cdot 2500 \cdot 1 = 20 + 2500 = \boxed{2520}$.

   Proposed by Matthew Qian

3. **Problem:** Given that numbers a, b, c satisfy $a + b + c = 0$, $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 10$, and $ab + bc + ac \neq 0$, compute the value of $\frac{-a^2 - b^2 - c^2}{ab + bc + ac}$.

   Solution:
   Note that $(a + b + c)^2 - 2(ab + bc + ac) = a^2 + b^2 + c^2 + 2(ab + bc + ac)$. We are given $a + b + c = 0$, so $-a^2 - b^2 - c^2 = 2(ab + bc + ac)$. Thus, $\frac{-a^2 - b^2 - c^2}{ab + bc + ac} = \boxed{2}$.

   Proposed by Jerry Li

Round 5

1. **Problem:** A circle with a radius of 1 is inscribed in a regular hexagon. This hexagon is inscribed in a larger circle. If the area that is outside the hexagon but inside the larger circle can be expressed as $\frac{a\pi}{b} - c\sqrt{d}$, where a, b, c, d are positive integers, a, b are relatively prime, and no prime perfect square divides into d. find the value of $a + b + c + d$.

   Solution: If the radius of the inscribed circle is 1, then the apothem of the hexagon is 1. We see that half the side of the hexagon, the radius of the inscribed circle and the radius of the circumscribed circle form a $30 - 60 - 90$ triangle. Using $30 - 60 - 90$ laws we can conclude that the side of the hexagon is $\frac{2\sqrt{3}}{3}$. Thus, the area of the hexagon is $2\sqrt{3}$. Using the $30 - 60 - 90$ triangle, we know that the radius of the circumscribed circle is $\frac{2\sqrt{3}}{3}$ which means that is area is $\frac{4\pi}{3}$. Subtracting the area of the hexagon from the area of the circumscribed circle yields us with our desired area: $\frac{4\pi}{3} - 2\sqrt{3}$. Thus, the sum of a, b, c and d are $\boxed{12}$.

   Proposed by Lakshika Kamalaganesh

2. **Problem:** At a dinner party, 10 people are to be seated at a round table. If person A cannot be seated next to person B and person C must be next to person D, how many ways can the 10 people be seated? Consider rotations of a configuration identical.

Solution: Let us first deal with the restriction that person C and D must sit next to each other. Let's pretend as if C and D are on person. Accounting the fact that they can sit vice versa: the people can sit at a round table in $\frac{9!\times 2}{9}$ ways. Now let's deal with the other restriction that A cannot sit next to D by again imagining as if they were on person sitting at the round table accounting for them sitting vice versa. Now we see that there are $\frac{8!\times 2\times 2}{8}$ ways that they could sit next to each other. Subtracting this from the total number of ways that the people can be seated next to each other yields us with: $\frac{9!\times 2}{9} - \frac{8!\times 2\times 2}{8} = 7! \times 2 \times 6 = \boxed{60480}$.

Proposed by Alice Hui

3. **Problem:** Let N be the sum of all the positive integers that are less than 2022 and relatively prime to 1011. Find $\dfrac{N}{2022}$.

   Solution: Observe that for any number x less than 1011 that is relatively prime to it, $2022 - x$ is also relatively prime to 1011, and vice versa. This means all numbers less than 2022 relatively prime to 1011 can be paired up into pairs that sum to 2022. Since we divide the sum at the end by 2022, we're really just counting the number of such pairs, which is the number of numbers less than 1011 relatively prime to it. This can be calculated from the Euler totient function of 1011. Since the prime factorization of 1011 is $3 \cdot 337$, the Euler totient function of 1011 is $1011 \cdot \frac{2}{3} \cdot \frac{336}{337} = \boxed{672}$.

   Proposed by Alice Hui

Round 6

1. **Problem:** The line $y = m(x-6)$ passes through the point A (6, 0), and the line $y = 8 - \dfrac{x}{m}$ pass through point B (0, 8). The two lines intersect at point C. What is the largest possible area of triangle ABC?

   Solution: Note that $y = m(x-6)$ has slope m and $y = 8 - \dfrac{x}{m}$ has slope $-\dfrac{1}{m}$. Thus, the lines are perpendicular since the slopes multiply to -1, and $\triangle ABC$ is a right triangle at angle C. Furthermore, we know that $AB = 10$ so the right triangle has hypotenuse 10. In a right triangle with a fixed hypotenuse, to maximize the area we would set the leg lengths equal (proof below). Thus, if s is the length of a side, then $s^2 + s^2 = 100 \Rightarrow\ s = \sqrt{\dfrac{100}{2}}$. Consequently, the maximum is $s^2/2 = \boxed{25}$.

   Proof: Let l be the length of the hypotenuse, and x, y be the length of the two legs. We know that $x^2 + y^2 = l^2$. We want to maximize xy. Note that by QM-GM, we have $\sqrt{\dfrac{x^2+y^2}{2}} = \sqrt{\dfrac{l^2}{2}} \geq \sqrt{xy}$ where equality occurs when $x = y$. Squaring both sides yields $\dfrac{l^2}{2} \geq xy$. To maximize xy, we let $xy = \dfrac{l^2}{2}$ which occurs when $x = y$, and we are done.

   Proposed by Alice Hui

2. **Problem:** Let N be the number of ways there are to arrange the letters of the word *MATHEMATICAL* such that no two As can be adjacent. Find the last 3 digits of $\dfrac{N}{100}$.

   Solution: Let's first count the number of ways to permute the letters without the As. Since there are 9 letters left and there are 2 each of M and T, we have $\frac{9!}{2!2!} = 90720$ ways to do so. Now, we have to count ways to insert the As so they aren't adjacent. If we the imagine gaps between each of the 9 placed letters as well as outside on the left and right as "slots," we have 10 slots to be filled by 3 As. There are $\binom{10}{3} = 120$ ways to so, giving a total of $90720 \cdot 120 = 10886400$. Dividing this by 100 and taking the last 3 digits, we get $\boxed{864}$.

   Proposed by Lakshika Kamalaganesh

3. **Problem:** Find the number of ordered triples of integers (a, b, c) such that $|a|, |b|, |c| \leq 100$ and $3abc = a^3 + b^3 + c^3$.

   Solution:

   Note that $3abc = a^3 + b^3 + c^3 \Rightarrow a^3 + b^3 + c^3 - 3abc = 0$. Furthermore, $a^3 + b^3 + c^3 - 3abc$ factors as $(a+b+c)(a^2+b^2+c^2-ab-bc-ac) = \frac{1}{2}(a+b+c+)((a-b)^2+(b-c)^2+(a-c)^2) = 0$.

   Thus, either $a + b + c = 0$ or $(a-b)^2 + (b-c)^2 + (a-c)^2 = 0$. The latter is true only when $a = b = c$. Since a can take on values from -100 to 100 this yields a total of $100 - (-100) + 1 = 201$ ordered triples of integers. For the former to be true, we do casework on the value of a. We need $b + c = -a$. The condition that $|a|, |b|, |c| \leq 100$ limits the number of ordered triples there are.

   If $a = -100$, then $b + c = 100$. We see that b can range from 100 to 0 and there will be a value for c such that $|c| \leq 100$. This case yields 101 triples. If $a = -99$, then $b + c = 99$. We see that b can range from 100 to -1 and there will be a solution for c. This case yields 102 ordered triples

   This goes on to when $a = 0$. We have $b + c = 0$ so b can range from 100 to -100, for 201 ordered triples.

   Then, if $a = 1$, we must have $b + c = -1$. Then, b can range from 99 to -100 for 200 ordered triples.

   This goes to when $a = 100$, and we must have $b + c = -100$. Then, b can range from 0 to -100 for 101 ordered triples.

   Hence, there are $101 + 102 + 103 + \cdots + 201 + 200 + 199 + \ldots 102 + 101$ total ordered triples for when $a + b + c = 0$. We sum this as $201 + 2(101 + 102 + \cdots + 200) = 201 + 2 \cdot 100 \cdot (101 + 200)/2 = 30301$ ordered triples.

   Summing the cases up yields $201 + 30301 = 30502$ ordered triples. However, we must be careful for overlap; the case $(0, 0, 0)$ is counted twice (and is the only triple overcounted), so we subtract 1 from our count for $\boxed{30501}$ ordered triples.

   Proposed by Jerry Li

Round 7

1. **Problem:** In a given plane, let A, B be points such that $AB = 6$. Let S be the set of points such that for any point C in S, the circumradius of $\triangle ABC$ is at most 6. Find $a + b + c$ if the area of S can be expressed as $a\pi + b\sqrt{c}$ where a, b, c are positive integers, and c is not divisible by the square of any prime.

   Solution: Let O_1, O_2 be the two distinct points such that triangles $O1AB$ and $O2AB$ are equilateral. Let M be the midpoint of AB. Denote by O the circumradius of $\triangle ABC$. Note that O lies on the perpendicular bisector of segment AB, namely, O lies on line O_1O_2. Thus, $C \in S$ if and only if O lies on segment O_1O_2.

   Define Ω_1, Ω_2 to be the circumcirlces of $O1AB$ and $O2AB$, respectively.

   We claim S is the region inside exactly one of Ω_1, Ω_2.

   First, any point in S, we know O lies on segment O_1O_2. It is clear that for any O lying on segment O_1O_2, all points on the circumcircle of $\triangle ABC$ lie inside exactly one of Ω_1, Ω_2 (Equivalently, all points outside both or inside both Ω_1, Ω_2 have circumcenters outside segment O_1O_2.) Thus, all points in S lie inside exactly one of Ω_1, Ω_2.

   For the other direction, consider a point C inside Ω_1 but outside Ω_2. View the value of $CP - AP$ as a continuous function of P, where P is some point on segment O_1O_2. For $P = O_1$, we have $CP < 6 = AP$ and for $P = O_2$, we have $CP > 6 = AP$. Then by the Intermediate Value Theorem, there exists P on segment O_1O_2 with $CP - AP = 0$, implying that O lies on segment O_1O_2. Analogous arguments apply for C inside Ω_2 but outside Ω_1. This proves our claim.

Finally, we compute the area of S by taking twice the area of Ω_1 and subtracting twice the area of the intersection of Ω_1 and Ω_2. The intersection is composed of two pieces which each have area equal to the area of sector O_1AB and equilateral triangle O_1AB. Then we get

$$2 \cdot 36\pi - 2\left(2\left(\frac{1}{6} \cdot 36\pi - \frac{6^2\sqrt{3}}{4}\right)\right) = 48\pi + 36\sqrt{3},$$

for a final answer of $\boxed{87}$.

Proposed by Jerry Li

2. **Problem:** Compute $\sum_{1 \le a < b < c \le 7} abc$.

   Solution:

   To find this sum, we first find $A = \sum_{1 \le a,b,c \le 7 \text{ and } a \ne b, b \ne c, a \ne c} abc$. Then, $S/6 = B = \sum_{1 \le a < b < c \le 7} abc$, since B is one of 6 identical sums, each sum with different orders when a, b, c is arranged in increasing order of values.

   To find A, we would want to simply multiply

   $$(1+2+3+4+5+6+7)(1+2+3+4+5+6+7)(1+2+3+4+5+6+7) = (1+2+3+4+5+6+7)^3$$

   where we take a from the first sum, b from the second sum, and c from the third sum. However, when expanded this product has terms in the form abc where $a = b, b = c$, or $a = c$. We want to exclude such terms, so we subtract the sum of all such terms. The sum of all products abc where $a = b$ would be

   $$(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2)(1 + 2 + 3 + 4 + 5 + 6 + 7).$$

   We need to subtract this quantity 3 times from $(1 + 2 + 3 + 4 + 5 + 6 + 7)$, once for $a = b$, once for $b = c$, once for $a = c$. Thus, our current sum appears to be

   $$(1 + 2 + 3 + 4 + 5 + 6 + 7)^3 - 3(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2)(1 + 2 + 3 + 4 + 5 + 6 + 7).$$

   However, note that the terms in the form of abc where $a = b = c$ are counted once in $(1 + 2 + 3 + 4 + 5 + 6 + 7)^3$ but subtracted thrice in

   $$3(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2)(1 + 2 + 3 + 4 + 5 + 6 + 7).$$

   We want to count these terms a total of 0 times, so we must add terms in the form abc where $a = b = c$ twice. Note that all the terms in the form abc where $a = b = c$ sum to $(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3)$. Hence, $S = A/6 = \frac{(1+2+3+4+5+6+7)^3 - 3(1^2+2^2+3^2+4^2+5^2+6^2+7^2)(1+2+3+4+5+6+7) + 2(1^3+2^3+3^3+4^3+5^3+6^3+7^3)}{6} = \boxed{1960}$, where we can apply the sum of squares and sum of cubes formula to evaluate the mess.

   Proposed by Jerry Li

3. **Problem:** Three identical circles are centered at points A, B, and C respectively and are drawn inside a unit circle. The circles are internally tangent to the unit circle and externally tangent to each other. A circle centered at point D is externally tangent to circles A, B, and C. If a circle centered at point E is externally tangent to circles A, B, and D, what is the radius of circle E? The radius of circle E can be expressed as $\frac{a\sqrt{b}-c}{d}$ where a, b, c, and d are all positive integers, and $\gcd(a, c, d) = 1$ and b is not divisible by the square of any prime. What is the sum of $a + b + c + d$?

   Solution: We find each circle's radius, approaching circle E with each radius found. To find the radius of A, B, C, let the radius of these circle be r. Then, through connecting centers and points of tangency, we can obtain the equation $r + \frac{2r}{\sqrt{3}} = 1 \Rightarrow r = 2\sqrt{3} - 3$. Let the radius of D be r_D. Then, the radius $r_D = 1 - 2(2\sqrt{3}-3) = 7 - 4\sqrt{3}$, since the center of D is the same as the center of the circle with radius 1. To find the radius of E. Let the radius of E be r_E. Let us refer to the centers of the circle with the circle name (so center of circle A is just A). We take a look at $\triangle DAB$ more carefully. Note that $DA = 1 - (2\sqrt{3}-3) = 4 - 2\sqrt{3}$. Let the altitude from D to AB be F. Note that F is also the point of tangency between circles A and B. We easily find $DF = \sqrt{(4-2\sqrt{3})^2 - (2\sqrt{3}-3)^2}$. Additionally, $DF = DE + DF = 7 - 4\sqrt{3} + r_E + DF$. We know that $DF = \sqrt{(2\sqrt{3}-3+r_E)^2 - (2\sqrt{3}-3)^2}$ by Pythagoras. Thus,

$$DF = \sqrt{(4-2\sqrt{3})^2 - (2\sqrt{3}-3)^2} = 7 - 4\sqrt{3} + r_E + \sqrt{(2\sqrt{3}-3+r_E)^2 - (2\sqrt{3}-3)^2}$$

While the equation looks very nasty, fortunately $\sqrt{(4-2\sqrt{3})^2 - (2\sqrt{3}-3)^2} = \sqrt{16+12-16\sqrt{3}-12-9+12\sqrt{3}} = \sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$. Hence,

$$3\sqrt{3} - 5 - r_E = \sqrt{(2\sqrt{3}-3+r_E)^2 - (2\sqrt{3}-3)^2}.$$

Squaring both sides and simplifying, we find $52 - 30\sqrt{3} = 10\sqrt{3}r - 16r \Rightarrow r = \frac{10\sqrt{3}-17}{11} \Rightarrow a+b+c+d = \boxed{41}$.

Proposed by Alice Hui

Round 8

Problem: Let A be the number of unused Algebra problems in our problem bank. Let B be the number of times the letter 'b' appears in our problem bank. Let M be the median speed round score. Finally, let C be the number of correct answers to Speed Round Question 1. Estimate

$$A \cdot B + M \cdot C.$$

Your answer will be scored according to the following formula, where X is the correct answer and I is your input.

$$\max\left\{0, \left\lceil \min\left\{13 - \frac{|I-X|}{0.05|I|}, 13 - \frac{|I-X|}{0.05|I-2X|}\right\}\right\rceil\right\}.$$

Solution:
A = 60 B = 1736 times (452 in algebra, 555 times in combo, 414 times in geo, 225 times in nt)

Proposed by Jerry Tan