Acton-Boxborough Math Competition 2021 Solutions

ABMC Team

April 23, 2021

Speed Round

1. **Problem:** You and nine friends spend 4000 dollars on tickets to attend the new Harry Styles concert. Unfortunately, six friends cancel last minute due to the flu. You and your remaining friends still attend the concert and split the original cost of 4000 dollars equally. What percent of the total cost does each remaining individual have to pay?

   Solution: At the start, there were ten people total. However, after six friends cancel, we are left with 4 people total. Each individual would then have to pay $\frac{1}{4} = \boxed{25}\%$ of the total.

   Proposed by Anusha Senapati

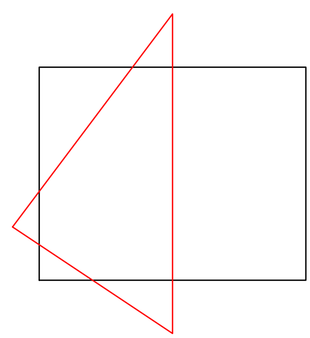
2. **Problem:** Find the number distinct 4 digit numbers that can be formed by arranging the digits of 2021.

   Solution: The only times in which one cannot make a distinct 4 digit number is when 0 is the leading digit, so we use complementary counting. There are $\frac{4!}{2} = 12$ ways to rearrange the digits without restrictions. There are $\frac{3!}{2} = 3$ ways to rearrange the digits with a leading digit of 0. Thus, there are $12 - 3 = \boxed{9}$ numbers that can formed.

   Proposed by Jerry Li

3. **Problem:** On a plane, Darnay draws a triangle and a rectangle such that each side of the triangle intersects each side of the rectangle at no more than one point. What is the largest possible number of points of intersection of the two shapes?

   Solution: It is clear that any given side of the triangle intersects the rectangle at most twice. A triangle has 3 sides, so the maximum number of points of intersection is then $3 \cdot 2 = \boxed{6}$, which is achievable.

   Proposed by Jerry Tan

4. **Problem:** Joy is thinking of a two-digit number. Her hint is that her number is the sum of two 2-digit perfect squares x_1 and x_2 such that exactly one of $x_i - 1$ and $x_i + 1$ is prime for each $i = 1, 2$. What is Joy's number?

   Solution: We list all 2-digit perfect squares to begin: $16, 25, 36, 49, 64, 81$.

   Now we must check them each for required conditions. First for 16, we have $16 - 1 = 15$ and $16 + 1 = 17$. Because 15 is not prime but 17 is, 16 satisfies the conditions. Continuing similarly, we check all of the possibilities to find the only other satisfying number is 36. Thus, Joy's number is the sum $16 + 36 = \boxed{52}$.

   Proposed by Annie Wang

5. **Problem:** At the North Pole, ice tends to grow in parallelogram structures of area 60. On the other hand, at the South Pole, ice grows in right triangular structures, in which each triangular and parallelogram structure have the same area. If every ice triangle ABC has legs $\overline{AB}$ and $\overline{AC}$ that are integer lengths, how many distinct possible lengths are there for the hypotenuse $\overline{BC}$?

   Solution: Since the right triangles at the south pole have the same area as the parallelograms at the north pole, we know they must also have area 60. Thus if we take a and b to be the lengths of the legs of the triangle, we have $\frac{ab}{2} = 60$, implying that $ab = 120$. Since a, b are positive integers we consider the factors of 120. We get the pairs $(1, 120), (2, 60), (3, 40), (4, 30), (5, 24), (6, 20), (8, 15), (10, 12)$ which we take without order since swapping the values of a and b does not change the length of the hypotenuse. We can quickly verify that each of these pairs produces a different hypotenuse length, giving a total of $\boxed{8}$ possible values.

   Proposed by Annie Wang

6. **Problem:** Carlsen has some squares and equilateral triangles, all of side length 1. When he adds up the interior angles of all shapes, he gets 1800°. When he adds up the perimeters of all shapes, he gets 24. How many squares does he have?

   Solution: Suppose first that we have x squares and y triangles. Note that the angles of a square add to 360° and the angles of a triangle add to 180°, giving the equation $360x + 180y = 1800$, which simplifies to $2x + y = 10$. In addition we have $4x + 3y = 24$ since we know the sum of the perimeters. Solving the linear system of equations for x and y gives $x = 3$ and $y = 4$, so there are a total of $\boxed{3}$ squares.

   Proposed by Jerry Tan

7. **Problem:** Vijay wants to hide his gold bars by melting and mixing them into a water bottle. He adds 100 grams of liquid gold to 100 grams of water. His liquefied gold bars have a density of 20 g/ml and water has a density of 1 g/ml. Given that the density of the mixture in g/mL can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, compute the sum $m + n$. (Note: density is mass divided by volume, gram (g) is unit of mass and ml is unit of volume. Further, assume the volume of the mixture is the sum of the volumes of the components.)

   Solution: First we want to compute the volume of each component before they are mixed. We have $V_{gold} = 100g \cdot \frac{1mL}{20g} = 5mL$, and $V_{water} = 100g \cdot \frac{1ml}{1g} = 100mL$. Therefore the total volume is $V_{total} = V_{gold} + V_{water} = 105mL$. Now we can calculate the density using the formula $\rho = \frac{m_{total}}{V_{total}} = \frac{200g}{105mL} = \frac{40}{21}$. Therefore we have $m + n = 40 + 21 = \boxed{61}$.

   Proposed by Jerry Tan

8. **Problem:** Julius Caesar has epilepsy. Specifically, if he sees 3 or more flashes of light within a 0.1 second time frame, he will have a seizure. His enemy Brutus has imprisoned him in a room with 4 screens, which flash exactly every 4, 5, 6, and 7 seconds, respectively. The screens all flash at once, and 105 seconds later, Caesar opens his eyes. How many seconds after he opened his eyes will Caesar first get a seizure?

   Solution: We want to figure out the smallest integer bigger than 105 which is a multiple of 3 or more of $\{4, 5, 6, 7\}$. To start, check the least common multiple of each three element subset: $\text{lcm}(4, 5, 6) = 60$, $\text{lcm}(4, 5, 7) = 140$, $\text{lcm}(4, 6, 7) = 84$, $\text{lcm}(5, 6, 7) = 210$. We see that $2\,\text{lcm}(4, 5, 6) = 120$ is the least multiple of any three that is bigger than 105. Therefore Caesar will first have a seizure $\boxed{15}$ seconds after opening his eyes.

   Proposed by Jerry Tan

9. **Problem:** Angela has a large collection of glass statues. One day, she was bored and decided to use some of her statues to create an entirely new one. She melted a sphere with radius 12 and a cone with

height of 18 and base radius of 2. If Angela wishes to create a new cone with a base radius 2, what would the the height of the newly created cone be?

Solution: The first part to solving this problem would be to find the total volume of the statues she is choosing to melt. To do this, we can use the formula for the volume of a sphere: $\frac{4}{3}\pi r^3$ and the volume of a cone: $\frac{h}{3}\pi r^2$. The two sphere with radius 12 has volume 2304π. Next, calculate the volume of the cone to get 24π. Adding all of these values together gives a sum: $2304\pi + 24\pi = 2328\pi$. If the new cone that Angela wishes to create must have a base radius of 2, then we can work backwards to find that $\frac{h}{3}\pi \cdot 2^2 = 2328\pi$, so $h = \boxed{1746}$.

Proposed by Anusha Senapati

10. **Problem:** Find the smallest positive integer N satisfying these properties:

    (a) No perfect square besides 1 divides N.

    (b) N has exactly 16 positive integer factors.

    Solution: The first condition implies that the power of each prime in N must be 1. Let there be x primes. The number of factors is obviously $(1+1)(1+1)\cdots(1+1)$ where $(1+1)$ is multiplied x times. Thus, the number of factors is 2^x. Since N has 16 factors we know $x = 4$. To minimize N we choose the smallest primes 4 primes to be in the prime factorization of N. The 4 smallest primes are $2, 3, 5, 7$ so $N = 2 \cdot 3 \cdot 5 \cdot 7 = \boxed{210}$.

    Proposed by Jerry Li

11. **Problem:** The probability of a basketball player making a free throw is $\frac{1}{5}$. The probability that she gets exactly 2 out of 4 free throws in her next game can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

    Solution: The player needs to make 2 free throws and miss 2 free throws. Each is independent, so this probability is $(\frac{1}{5})(\frac{1}{5})(1-\frac{1}{5})(1-\frac{1}{5}) = \frac{16}{625}$. Then, because these shots can be made and missed in any order, we multiply by the ways to order them, which is $\binom{4}{2} = 6$. Thus, the probability of making exactly 2 in 4 free throws is $6 \cdot \frac{16}{625} = \frac{96}{625}$. The final answer in correct form is $96 + 625 = \boxed{721}$.

    Proposed by Anusha Senapati

12. **Problem:** A new donut shop has 1000 boxes of donuts and 1000 customers arriving. The boxes are numbered 1 to 1000. Initially, all boxes are lined up by increasing numbering and closed. On the first day of opening, the first customer enters the shop and opens all the boxes for taste testing. On the second day of opening, the second customer enters and closes every box with an even number. The third customer then "reverses" (if closed, they open it and if open, they close it) every box numbered with a multiple of three, and so on, until all 1000 customers get kicked out for having entered the shop and reversing their set of boxes. What is the number on the sixth box that is left open?

    Solution: Observe that a box will be closed if it has an even number of factors, and open if it has an odd number of factors. Since a number has an odd number of factors if and only if it is a perfect square, the only boxes that are opened will be the boxes numbered with perfect squares. Therefore the 6th box will be numbered $6^2 = \boxed{36}$.

    Proposed by Anusha Senapati

13. **Problem:** For an assignment in his math class, Michael must stare at an analog clock for a period of 7 hours. He must record the times at which the minute hand and hour hand form an angle of exactly 90°, and he will receive 1 point for every time he records correctly. What is the maximum number of points Michael can earn on his assignment?

Solution: Consider the relative speeds of the hands. Since the minute hand moves 12 times faster than the hour hand, this means that the minute hand travels 330° relative to the hour hand in 1 hour. Call a cycle the amount it takes the hands to go from overlapping, to overlapping again, i.e. the minute hand is 360° ahead of the hour hand relative to the start. So one cycle takes $\frac{360}{330} = \frac{12}{11}$ hours to complete. Note that in every cycle, the hour and minute hand make a right angle twice: once at 90° and 270°. The total number of cycles we can fit into 7 hours is $\frac{7}{\frac{12}{11}} = \frac{77}{12} = 6\frac{5}{12}$ cycles. During the 6 completed cycles the hands must make the desired angle a total of 12 times. In the leftover $\frac{5}{12}$ of a cycle we can fit a maximum of 1 further occurrence, since to include the end of a cycle at the start, and the beginning of a cycle at the end, there would have to be more than $\frac{1}{2}$ of a cycle remaining. This gives a maximum of $\boxed{13}$ points.

Proposed by Jerry Tan

14. **Problem:** The graphs of $y = x^3 + 5x^2 + 4x - 3$ and $y = -\dfrac{1}{5}x + 1$ intersect at three points in the Cartesian plane. Find the sum of the y-coordinates of these three points.

    Solution: We set the two equations equal to get $-\dfrac{1}{5}x + 1 = x^3 + 5x^2 + 4x - 3$, which becomes

    $$x^3 + 5x^2 + \frac{21}{5}x - 4 = 0.$$

    We are given three intersection points which means this cubic has three real solutions. If the three intersection points are $(x_1, y_2); (x_2, y_2); (x_3, y_3)$, then $x_1 + x_2 + x_3 = \dfrac{-5}{1} = -5$ by Vieta's. Summing the line equation over the three points, we find $y_1 + y_2 + y_3 = -\dfrac{1}{5}(x_1 + x_2 + x_3) + 3(1) = \boxed{4}$.

    Proposed by Jerry Tan

15. **Problem:** In the quarterfinals of a single elimination countdown competition, the 8 competitors are all of equal skill. When any 2 of them compete, there is exactly a 50% chance of either one winning. If the initial bracket is randomized, the probability that two of the competitors, Daniel and Anish, face off in one of the rounds can be expressed as $\dfrac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

    Solution: Label the four pairs that compete in the quarterfinals with numbers 1 to 4 such that the winners of pairs 1 and 2 play in the semifinals and the winner of pairs 3 and 4 play. WLOG, Daniel is in pair 1. We do casework based on which round Daniel and Anish face off in.

    Case 1: They compete in quarterfinals. Then Anish must be in pair 1, which occurs with probability $\dfrac{1}{7}$.

    Case 2: They compete in semifinals. Then Anish must be in pair 2, and both Daniel and Anish must beat one opponent to reach the semifinals. Thus this occurs with probability $\dfrac{2}{7} \cdot \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{14}$.

    Case 3: They compete in finals. Then Anish must be in pair 3 or 4 and both Daniel and Anish must beat two opponent to reach the finals. Thus this occurs with probability $\dfrac{4}{7} \cdot \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{28}$.

    The total probability is then $\dfrac{1}{7} + \dfrac{1}{14} + \dfrac{1}{28} = \dfrac{1}{4}$. The answer is $\boxed{5}$.

    Proposed by Jerry Tan

16. **Problem:** How many positive integers less than or equal to 1000 are not divisible by any of the numbers $2, 3, 5$ and 11?

Solution: The number can be split into two parts; we just sum the answer to each part. The parts are as follows:

A) Number of positive integers less than or equal to 990 not divisible by $2, 3, 5, 11$ B) Number of positive integers greater than 990 not divisible by $2, 3, 5, 11$.

We know for A that $\phi(990) = 990 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{10}{11} = 240$ positive integers less than or equal to 990 are relatively prime to 990, which is equivalent to the desired condition. Additionally, B is relatively easy to count; we find only two numbers, namely $991, 997$. Thus, our answer is $240 + 2 = \boxed{242}$.

Proposed by Jerry Li

17. **Problem:** A strictly increasing geometric sequence of positive integers $a_1, a_2, a_3 \cdots$ satisfies the following properties:

    (a) Each term leaves a common remainder when divided by 7

    (b) The first term is an integer from 1 to 6

    (c) The common ratio is an perfect square

    Let N be the smallest possible value of $\frac{a_{2021}}{a_1}$. Find the remainder when N is divided by 100.

    Solution: Let the first number be a and the common ratio be r^2. Since every term of the sequence leaves a common remainder when divided by 7 we know that $a \equiv ar^2 \equiv ar^4 \pmod 7 \cdots$. Thus, we know that $r^2 \equiv 1 \pmod 7$ as any other residue would change the residues every time you multiplied a term by the common ratio. This simplifies to $r^2 - 1 \equiv (r+1)(r-1) \equiv 0 \pmod 7$. Thus, either $r \equiv 6 \pmod 7$ or $r \equiv 1 \pmod 7$. Since the term is increasing, we toss out $r = 1$ and consequently $r = 6$ is the smallest r value. Remembering that $r^2 = 36$ is the common ratio, we want $\frac{a_{2021}}{a_1} \pmod{100}$ which is $36^{2020} \pmod{100}$. By Chinese Remainder Theorem and Euler's totient, we find this to be $0 \pmod 4$ and $1 \pmod{25}$, so the answer is $\boxed{76} \pmod{100}$.

    Proposed by Jerry Li

18. **Problem:** Suppose $p(x) = x^3 - 11x^2 + 36x - 36$ has roots r, s, t. Find $\frac{r^2+s^2}{t} + \frac{s^2+t^2}{r} + \frac{t^2+r^2}{s}$.

    Solution: First we observe that $r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs+st+tr)$, which we know by vieta's formula is $(11)^2 - 2(36) = 49$. Therefore we can rewrite the desired quantity as $\frac{49-r^2}{r} + \frac{49-s^2}{s} + \frac{49-t^2}{t} = 49(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}) - (r+s+t)$. Rewriting again, we have $49(\frac{rs+st+tr}{rst}) - (r+s+t) = 49 \cdot \frac{36}{36} - 11 = \boxed{38}$.

    Alternatively, notice that the roots of p are $2, 3$, and 6, giving $\frac{2^2+3^2}{6} + \frac{3^2+6^2}{2} + \frac{6^2+2^2}{3} = \frac{13}{6} + \frac{45}{2} + \frac{40}{3} = 38$.

    Proposed by Devin Brown

19. **Problem:** Let $a, b \leq 2021$ be positive integers. Given that ab^2 and a^2b are both perfect squares, let $G = \gcd(a, b)$. Find the sum of all possible values of G.

    Solution: Since b^2 is obviously a perfect square, and ab^2 is a perfect square, a must be a perfect square since otherwise $a \cdot b^2$ wouldn't be a perfect square. Using similar logic for the fact that a^2b is a perfect square, we find b to be a perfect square. Let $a = m^2, b = n^2$. We know that $m^2 \leq 2021 \Rightarrow m \leq 44$ and similarly $n^2 \leq 2021 \Rightarrow n \leq 44$. It is well known that $\gcd(m^2, n^2) = \gcd(m, n)^2$. Since $\gcd(m, n)$ can take on all values from 1 to 44 we know that G is simply the sum of squares from 1^2 to 44^2 which, by the sum of squares formula, is $\frac{44 \cdot 45 \cdot 89}{6} = \boxed{29370}$.

    Proposed by Jerry Li

20. **Problem:** Jessica rolls six fair standard six-sided dice at the same time. Given that she rolled at least four 2's and exactly one 3, the probability that all six dice display prime numbers can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. What is $m+n$?

Solution: This is conditional probability. We need the probability of rolling at least four 2's and exactly one 3.

If exactly four 2's are rolled, then if the sixth roll cannot be a 3 nor a 2, and the probability is $\left(\frac{1}{6}\right)^5 \frac{4}{6} \cdot \frac{6!}{4!} = \frac{120}{6^6}$ since there are 4 numbers we can choose from to complete the final roll $(1, 4, 5, 6)$ each in each case there are 30 ways to arrange the 6 numbers.

If five 2's are rolled, then the probability is $\left(\frac{1}{6}\right)^6 \frac{6!}{5!} = \frac{6}{\cdot 6^6}$.

The total probability of rolling at least four 2's and exactly one 3 is then $\frac{120}{6^6} + \frac{6}{6^6} = \frac{120+6}{6^6} = \frac{126}{6^6}$.

Now, the probability of all six primes given at least four 2's and exactly one 3 can be split into four 2's, one 3, and one 5, or five 2's, and one 3. The total is $\frac{30}{6^6} + \frac{6}{6^6} = \frac{36}{6^6}$. The desired probability is then $\frac{\frac{36}{6^6}}{\frac{126}{6^6}} = \frac{36}{126} = \frac{2}{7}$. Thus, $m+n = 2+7 = \boxed{9}$.

Proposed by Annie Wang

21. **Problem:** Let a, b, c be numbers such $a+b+c$ is real and the following equations hold:

$$a^3 + b^3 + c^3 = 25,$$
$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1,$$
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{25}{9}.$$

The value of $a+b+c$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m+n$.

Solution: Subtract $3abc$ from both sides of the first equation. We use the factorization $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac) = (a+b+c)((a+b+c)^2-3(ab+bc+ac))$. Therefore, we have

$$(a+b+c)((a+b+c)^2 - 3(ab+bc+ac)) = 25 - 3abc.$$

Now multiply the second and third equations by abc. We find that $a+b+c = abc$ and $ab+bc+ac = \frac{25abc}{9}$. Note that it suffices to find the value of abc since $abc = a+b+c$. Thus, we write our main equation in terms of abc. The main equation becomes

$$abc \cdot \left((abc)^2 - \frac{25abc}{3}\right) = 25 - 3abc$$
$$(abc)^2 \cdot \left(abc - \frac{25}{3}\right) = 25 - 3abc$$
$$(abc)^2 \cdot (25 - 3abc) = -3(25 - 3abc)$$

In order for this equation to be true, we can either have $(abc)^2 = -3$ or $(25-3abc) = 0$. The first case yields no real solutions. The second case yields $abc = \frac{25}{3}$ as a solution. The final answer is $\boxed{28}$.

Proposed by Jerry Li

22. **Problem:** Let ω be a circle and P be a point outside ω. Let line ℓ pass through P and intersect ω at points A, B and with $PA < PB$ and let m be another line passing through P intersecting ω at points C, D with $PC < PD$. Let X be the intersection of AD and BC. Given that $\frac{PC}{CD} = \frac{2}{3}, \frac{PC}{PA} = \frac{4}{5}$, and $\frac{[ABC]}{[ACD]} = \frac{7}{9}$, the value of $\frac{[BXD]}{[BXA]}$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$. **Note: due to an error in this problem, this question was not graded.**

    Solution: Let $PC = 4x$. From the two length ratios we find $CD = 6x$ and $PA = 5x$. By power of a point, we find $PA * PB = PC * PD \Rightarrow AB = 3x$. We are dealing with ratios, so we can assume $[AXC] = 1$, as we can scale everything and still preserve the problem conditions. Let $[AXB] = b$. Clearly $\triangle AXB \sim \triangle CXD$ with ratio $\frac{3x}{6x} = \frac{1}{2}$. Then $[CXD] = 4b$. The given condition is then

    $$\frac{b+1}{4b+1} = \frac{7}{9} \Rightarrow 9b + 9 = 28b + 7 \Rightarrow b = \frac{2}{19}.$$

    Then

    $$\frac{[BXD]}{[BXA]} = \frac{DX}{XA} = \frac{[CXD]}{[CXA]} = 4b = \frac{8}{19}.$$

    The answer is $8 + 19 = \boxed{27}$.

    Proposed by Jerry Li

23. **Problem:** Define the operation $a \circ b = \frac{a^2 + 2ab + a - 12}{b}$. Given that $1 \circ (2 \circ (3 \circ (\cdots 2019 \circ (2020 \circ 2021)))...)$ can be expressed as $-\frac{a}{b}$ for some relatively prime positive integers a, b, compute $a + b$.

    Solution: Note that $3 \circ x = \frac{9 + 6x + 3 - 12}{x} = \frac{6x}{x} = 6$ for some arbitrary $x \neq 0$. It is easy to see that $4 \circ (\cdots 2019 \circ (2020 \circ 2021))$ is positive, so the desired expression simplifies to $1 \circ (2 \circ 6)) = 1 \circ 3 = -\frac{4}{3}$. The desired sum is $4 + 3 = \boxed{7}$.

    Proposed by Jerry Li

24. **Problem:** Find the largest integer $n \leq 2021$ for which $5^{n-3} \mid (n!)^4$.

    Solution: Let $v_p(n)$ denote the number of times p divides into n. It is well known that $v_p(n!) = \frac{n - s_p(n)}{p-1}$ where $s_p(n)$ represents the sum of the digits of n represented in base p. The condition $5^{n-3} | (n!)^4$ thus becomes $n - 3 \leq \frac{n - s_5(n)}{4} \cdot 4 \Rightarrow n - 3 \leq n - s_5(n) \Rightarrow s_5(n) \leq 3$. Thus, we are looking for the largest n such that the sum of the digits of n in base 5 is less than or equal to 3 and $n \leq 2021$. The desired number is $30000_5 = \boxed{1875}$.

    Proposed by Jerry Li

25. **Problem:** On the Cartesian plane, a line ℓ intersects a parabola with a vertical axis of symmetry at $(0, 5)$ and $(4, 4)$. The focus F of the parabola lies below ℓ, and the distance from F to ℓ is $\frac{16}{\sqrt{17}}$. Let the vertex of the parabola be (x, y). The sum of all possible values of y can be expressed as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

    Solution: Let the focus have coordinates (x, y_1) and the directrix be $y = c$. By drawing a right triangle flush against ℓ with the long leg of length $\frac{16}{\sqrt{17}}$, we note that F must lie on a line of slope $-\frac{1}{4}$ with y-intercept $\frac{\sqrt{17}}{4} \cdot \frac{16}{\sqrt{17}} = 4$ less than that of ℓ. Thus we have

$$y_1 = -\frac{1}{4}x + 1 \quad (*).$$

By definition, each point on the parabola is equidistant from the focus and directrix, so

$$\sqrt{(4-c)^2} = \sqrt{(4-x)^2 + (4-y_1)^2}$$
$$\sqrt{(5-c)^2} \quad (1) = \sqrt{x^2 + (5-y_1)^2}.$$

Squaring and subtracting the equations yields

$$2c - 9 = 2y_1 + 7 - 8x \Rightarrow \ c = y_1 + 8 - 4x.$$

We can substitute x from $(*)$ to get $c = 17y_1 - 8$. Now we substitute all variables in (1) in terms of y_1. Simplifying, we get

$$17y_1^2 - 25y_1 + 8 = 0 \Rightarrow \ (17y_1 - 8)(y_1 - 1) = 0.$$

Thus $y_1 = 1, \frac{8}{17}$. Lastly, the vertex is equidistant from the focus and directrix, we have $y = 9y_1 - 4$. Then the possible values of y are $5, \frac{4}{17}$, and it is not hard to verify both yield valid parabolas. The answer is $89 + 17 = \boxed{106}$.

Proposed by Jerry Tan

Accuracy Round

1. **Problem:** There is a string of numbers 1234567891023 . . . 910134 . . . 91012 . . . that concatenates the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, then $2, 3, 4, 5, 6, 7, 8, 9, 10, 1$, then $3, 4, 5, 6, 7, 8, 9, 10, 1, 2$, and so on. After $10, 1, 2, 3, 4, 5, 6, 7, 8, 9$, the string will be concatenated with $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, again. What is the 2021st digit?

   Solution: After ten cycles of eleven digits, the digits will repeat. So the 2021st digit is the same as the $2021 - 18 * 110 = 41$st digit, which is the 8th digit of the fourth cycle, $4, 5, 6, 7, 8, 9, 10, 1, 2, 3$, which is $\boxed{0}$.

   Proposed by Annie Wang

2. **Problem:** Bob really likes eating rice. Bob starts eating at the rate of 1 bowl of rice per minute. Every minute, the number of bowls of rice Bob eats per minute increases by 1. Given there are 78 bowls of rice, find number of minutes Bob needs to finish all the rice.

   Solution: Bob eats $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ bowls of rice after n minutes. Thus, we are finding n such that $\dfrac{n(n+1)}{2} = 78$. The desired n is $\boxed{12}$.

   Proposed by Jerry Li

3. **Problem:** Suppose John has 4 fair coins, one red, one blue, one yellow, one green. If John flips all 4 coins at once, the probability he will land exactly 3 heads and land heads on both the blue and red coins can be expressed as $\dfrac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

   Solution: There is a $\dfrac{1}{4}$ chance of landing heads on both blue and red. John then needs at exactly one of yellow and green to be heads, which occurs with probability $\dfrac{2}{4}$. Thus the total probability is $\dfrac{1}{4} \cdot \dfrac{2}{4} = \dfrac{1}{8}$ for an answer of $\boxed{9}$.

   Proposed by Jerry Li

4. **Problem:** Three of the sides of an isosceles trapezoid have lengths $1, 10, 20$. Find the sum of all possible values of the fourth side.

   Solution: The two legs of an isosceles trapezoid are equal, so the fourth side can be $1, 10$ or 20. However, an isosceles trapezoid with bases 10 and 20 and legs 1 is impossible as it fails the triangle inequality. The answer is $10 + 20 = \boxed{30}$.

   Proposed by Jerry Tan

5. **Problem:** An number *two-three-delightful* if and only if it can be expressed as the product of 2 consecutive integers larger than 1 and as the product of 3 consecutive integers larger than 1. What is the smallest two-three-delightful number?

   Solution: We check $x(x+1)(x+2)$ for $x \geq 2$ to see if it is the product of 2 consecutive integers. Since $24, 60, 120$ all fail and $210 = 14 \cdot 15$, the answer is $\boxed{210}$.

   Proposed by Jerry Tan

6. **Problem:** There are 3 students total in Justin's online chemistry class. On a 100 point test, Justin's two classmates scored 4 and 7 points. The teacher notices that the class median score is equal to $\gcd(x, 42)$, where the positive integer x is Justin's score. Find the sum of all possible values of Justin's score.

Solution: The median score is $\begin{cases} 4 \text{ if } x \le 4 \\ x \text{ if } 4 < x < 7 \\ 7 \text{ if } x \ge 7 \end{cases}$. We can eliminate $x \le 4$ since 4 does not divide 42. For $4 < x < 7$, we see that $x = 6$ indeed works. Lastly, if $x \ge 7$, we only need $\gcd(x, 42) = 7$ which means x is a multiple of 7 relatively prime to 6. Then the possible values are $x = 7, 35, 49, 77, 91$, so the answer is $6 + 7 + 35 + 49 + 77 + 91 = \boxed{265}$.

Proposed by Jerry Tan

7. **Problem:** Eddie's gym class of 10 students decides to play ping pong. However, there are only 4 tables and only 2 people can play at a table. If 8 students are randomly selected to play and randomly assigned a partner to play against at a table, the probability that Eddie plays against Allen is $\frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

   Solution: Eddie has an $\frac{8}{10}$ of being selected to play. Then, there are 9 spots left, 1 of which is playing against Eddie. So, Allen has a $\frac{1}{9}$ chance of being assigned against Eddie. The total probability if $\frac{8}{10} \cdot \frac{1}{9} = \frac{4}{45}$. The answer is $\boxed{49}$.

   Proposed by Jerry Tan

8. **Problem:** Let S be the set of integers k consisting of nonzero digits, such that $300 < k < 400$ and $k - 300$ is not divisible by 11. For each k in S, let $A(k)$ denote the set of integers in S not equal to k that can be formed by permuting the digits of k. Find the number of integers k in S such that k is relatively prime to all elements of $A(k)$.

   Solution: The elements of S are of the form $3\overline{XY}$ where X, Y are distinct nonzero digits. Thus for $k = 3\overline{XY}$, $A(k) = \{3\overline{YX}\}$. WLOG take $X > Y$, then $\gcd(3\overline{XY}, 3\overline{YX}) = \gcd(300 + 10X + Y, 300 + 10Y + X) = \gcd(9(X - Y), 300 + 10Y + X)$. We will find ordered pairs (X, Y) such that the gcd is greater than 1. Note that $9|3\overline{YX} \Rightarrow 3|3\overline{YX}$ and $X - Y \le 8$ so we just need to check divisibility of $3\overline{YX}$ by $2, 3, 5$, and 7. Then 2 is a common divisor if X and Y are both even, which is for $4 \cdot 4 - 4 = 12$ pairs (X, Y).

   Next, 3 is a common divisor if $3|X + Y$, which holds for 24 pairs of (X, Y) corresponding to when $\overline{XY}$ is a multiple of 3 between 12 and 99 excluding $33, 66, 99$. However, this double counts pairs of even numbers (X, Y) with sum a multiple of three, namely $24, 42, 48, 84$, so we must subtract 4 pairs.

   5 is a common divisor when $X, Y \in \{0, 5\}$, which is impossible.

   Finally, 7 is a common divisor requires $X - Y = 7$, and check 381 and 392 only 392 is a multiple of 7. This gives 2 additional ordered pairs (X, Y).

   In total, $12 + 24 - 4 + 2 = 34$ pairs of (X, Y) fail the condition, and since there are $100 - 10 - 18 = 72$ elements in S total, the answer is $72 - 34 = \boxed{38}$.

   Proposed by Jerry Tan

9. **Problem:** In $\triangle ABC$, $AB = 6$ and $BC = 5$. Point D is on side AC such that BD bisects angle $\angle ABC$. Let E be the foot of the altitude from D to AB. Given $BE = 4$, find AC^2.

   Solution: Let F be the foot of the altitude from B to $\overline{AC}$. Let $AD = 6x$, so $CD = 5x$ and $AC = 11x$ by the Angle Bisector Theorem. We know $AE = 2$, so by the Pythagorean Theorem, $DE = \sqrt{36x^2 - 4}$. Since $\triangle ADE \sim \triangle ABF$, we have

$$\frac{BF}{DE} = \frac{FA}{EA} = \frac{AB}{AD} = \frac{1}{x},$$

so $BF = \dfrac{\sqrt{36x^2-4}}{x}$ and $FA = \dfrac{2}{x}$. Then, $CF = 11x - \dfrac{2}{x}$ and by Pythagorean Theorem we get

$$25 = \frac{36x^2-4}{x^2} + \frac{(11x^2-2)^2}{x^2} \Rightarrow\ 25x^2 = 36x^2 - 4 + 121x^4 - 44x^2 + 4 \Rightarrow\ 11x^2(11x^2-3) = 0,$$

so $x^2 = \dfrac{3}{11}$. Finally, $AC^2 = 121x^2 = \boxed{33}$.

Proposed by Jerry Tan

10. **Problem:** For each integer $1 \le n \le 10$, Abe writes the number $2^n + 1$ on a blackboard. Each minute, he takes two numbers a and b, erases them, and writes $\frac{ab-1}{a+b-2}$ instead. After 9 minutes, there is one number C left on the board. The minimum possible value of C can be expressed as $\dfrac{p}{q}$ for relatively prime positive integers p, q. Find $p+q$.

    Solution: Notice that $\dfrac{1}{\frac{ab-1}{a+b-2}-1} = \dfrac{1}{\frac{ab-a-b+1}{a+b-2}} = \dfrac{(a-1)+(b-1)}{(a-1)(b-1)} = \dfrac{1}{a-1} + \dfrac{1}{b-1}$. Thus, the sum over all numbers x on the board $\sum_x \dfrac{1}{x-1}$ is invariant (that is, stays the same constant value throughout the process.) In the beginning, $\sum_x \dfrac{1}{x-1} = \sum_{n=1}^{10} \dfrac{1}{2^n} = \dfrac{1023}{1024}$, so at the end of the 9 minutes we know $\dfrac{1}{C-1} = \dfrac{1023}{1024}$. Thus $C = \dfrac{2047}{1023}$ and the answer is $\boxed{3070}$.

    Proposed by Karthik Seetharaman

11. **Problem:** **Estimation:** Let A and B be the proportions of contestants that correctly answered Questions 9 and 10 of this round, respectively. Estimate $\left\lfloor \dfrac{1}{(AB)^2} \right\rfloor$.

    Solution: $\dfrac{34}{194}$ students answered problem 9 correctly. $\dfrac{14}{194}$ students answered problem 10 correctly. $\left\lfloor \dfrac{1}{(AB)^2} \right\rfloor = \boxed{6251}$.

    Proposed by Devin Brown

Team Round

Round 1

1. **Problem:** There are 99 dogs sitting in a long line. Starting with the third dog in the line, if every third dog barks three times, and all the other dogs each bark once, how many barks are there in total?

   Solution: The first three dogs have a total of 5 barks. Similarly, the 4th, 5th, and 6th dogs have 5 total barks. There are $99/3 = 33$ such groups, so there are $33 \cdot 5 = \boxed{165}$ barks in total.

   Proposed by Annie Wang

2. **Problem:** Indigo notices that when she uses her lucky pencil, her test scores are always $66\frac{2}{3}\%$ higher than when she uses normal pencils. What percent lower is her test score when using a normal pencil than her test score when using her lucky pencil?

   Solution:

   Let her usual test score be x. Since she scores $66\frac{2}{3}\%$ higher, we know that her scores using her lucky pencil must be $\dfrac{5}{3}x$. Therefore, her test scores using a normal pencil would be $\dfrac{\frac{5}{3}x - x}{\frac{5}{3}x} \cdot 100 = \boxed{40}$ percent lower.

   Proposed by Jerry Tan

3. **Problem:** Bill has a farm with deer, sheep, and apple trees. He mostly enjoys looking after his apple trees, but somehow, the deer and sheep always want to eat the trees' leaves, so Bill decides to build a fence around his trees. The 60 trees are arranged in a 5×12 rectangular array with 5 feet between each pair of adjacent trees. If the rectangular fence is constructed 6 feet away from the array of trees, what is the area the fence encompasses in feet squared? (Ignore the width of the trees.)

   Solution: Each side will be extended 6 feet on both sides of the $5x12$ rectangular array, which is $4 \cdot 5$ by $11 \cdot 5$ or 20 by 55. Thus, the dimensions of the fence will be $(20+6+6)(55+6+6) = 32 \cdot 67 = \boxed{2144}$.

   Proposed by Annie Wang

Round 2

1. **Problem:** If x + 3y = 2, then what is the value of the expression: $9^x * 729^y$?

   Solution: We can rewrite the expression as $9^x * (9^3)^y = 9^x * 9^{3y} = 9^{x+3y}$ by the rules of exponentiation. Substituting, we find that the answer is $9^2 = \boxed{81}$.

   Proposed by Anusha Senapati

2. **Problem:** Lazy Sheep loves sleeping in, but unfortunately, he has school two days a week. If Lazy Sheep wakes up each day before school's starting time with probability $1/8$ independent of previous days, then the probability that Lazy Sheep wakes up late on at least one school day over a given week is $\dfrac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

   Solution: Approach this problem with complementary counting. The probability that Lazy Sheep wakes before the school start time on both school days is $\frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$. Hence, the probability he wakes up late on at least one school day is $1 - \frac{1}{64} = \frac{63}{64}$. The answer is thus $63 + 64 = \boxed{127}$.

   Proposed by Annie Wang

3. **Problem:** An integer n leaves remainder 1 when divided by 4. Find the sum of the possible remainders n leaves when divided by 20.

   Solution:

   Let $n = 20k + r$ where r is the remainder when n is divided by 20. Clearly, the remainder of $20k + r$ when divided by 4 is simply the remainder of r when divided by 4 since $20k$ is divisible by 4. Thus, we need r to leave a remainder of 1 when divided by 4. Furthermore, we know $0 \le r < 20$ since r is a remainder. The values of integral r such that these conditions hold are $1, 5, 9, 13, 17$ for a sum of $\boxed{45}$.

   Proposed by Jerry Li

Round 3

1. **Problem:** Jake has a circular knob with three settings that can freely rotate. Each minute, he rotates the knob $120°$ clockwise or counterclockwise at random. The probability that the knob is back in its original state after 4 minutes is $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

   Solution: The knob is back in its original state only if the knob has been rotated twice in one direction, and twice in another direction. Call a right turn r and a left turn l. The problem simplifies to the probability of choosing 4 slips of paper with r and l one at a time such that there are 2 rights and 2 lefts. There is $\binom{4}{2} = 6$ ways of choosing 2 rights and 2 lefts, and $2^4 = 16$ total possible ways one can choose the slips of paper. Thus the desired probability is $\frac{6}{16} = \frac{3}{8}$ so $p + q = \boxed{11}$.

   Proposed by Jerry Tan

2. **Problem:** Given that 3 not necessarily distinct primes p, q, r satisfy $p + 6q + 2r = 60$, find the sum of all possible values of $p + q + r$.

   Solution: Note that p must be even as $6q, 2r, 60$ are all even. Thus, p is both a prime and even, implying $p = 2$. Thus, we have $2 + 6q + 2r = 60 \Rightarrow 6q + 2r = 58 \Rightarrow 3q + r = 29 \Rightarrow r = 29 - 3q$. If q is even, then $q = 2, r = 23$ for the pair $(2, 2, 23)$ that works. Otherwise, if q is odd, we know r must be even since $29 - 3q$ is always even for odd q. Thus, if q is odd, we must have $r = 2$ and consequently $q = 9$, which is not prime. Therefore the only possible value S is $2 + 2 + 23 = \boxed{27}$.

   Proposed by Jerry Li

3. **Problem:** Dexter's favorite number is the positive integer x. If $15x$ has an even number of proper divisors, what is the smallest possible value of x? (Note: A proper divisor of a positive integer is a divisor other than itself.)

   Solution: If the number of proper divisors of $15x$ is even, then the number of divisors must be odd. A number has odd divisions if and only if the number is a perfect square. Thus, we must have $15x$ be a perfect square. The smallest such x such that $15x$ is a perfect square is $\boxed{15}$.

   Proposed by Jerry Tan

Round 4

1. **Problem:** Three circles of radius 1 are each tangent to the other two circles. A fourth circle is externally tangent to all three circles. The radius of the fourth circle can be expressed as $\frac{a\sqrt{b} - c}{d}$ for positive integers a, b, c, d where b is not divisible by the square of any prime and a and d are relatively prime. Find $a + b + c + d$.

Solution:

Note that the center of the fourth circle is the circumcenter of the triangle formed by connecting the centers of the other 3 circles. Note that we are looking for the circumradius minus the radius of one of the 3 circles, which is 1. The triangle formed by connecting the centers is just an equilateral triangle with side length 2. We compute the circumradius by using $30-60-90$ properties, and we find the circumradius to be $\frac{2}{\sqrt{3}}$. Thus, the desired answer is $\frac{2}{\sqrt{3}}-1=\frac{2\sqrt{3}-3}{3} \Rightarrow a+b+c+d=\boxed{11}$.

Proposed by Jerry Tan

2. **Problem:** Evaluate $\frac{\sqrt{15}}{3}\cdot\frac{\sqrt{35}}{5}\cdot\frac{\sqrt{63}}{7}\cdot\dots\cdot\frac{\sqrt{5475}}{73}$.

   Solution: "Unrationalize" the denominator for each term. The desired product becomes

$$\frac{\sqrt{5}}{\sqrt{3}}\cdot\frac{\sqrt{7}}{\sqrt{5}}\cdot\frac{\sqrt{9}}{\sqrt{7}}\cdot\dots\cdot\frac{\sqrt{75}}{\sqrt{73}}.$$

   This product clearly telescopes, or cancels a lot; all but the numerator of the last term and the denominator of the first term cancel, leaving a final product of $\frac{\sqrt{75}}{\sqrt{3}}=\sqrt{25}=\boxed{5}$.

   Proposed by Jerry Li

3. **Problem:** For any positive integer n, let $f(n)$ denote the number of digits in its base 10 representation, and let $g(n)$ denote the number of digits in its base 4 representation. For how many n is $g(n)$ an integer multiple of $f(n)$?

   Solution: For $f(n)=1$, which occurs for $n\in[1,9]$, all n will work. This contributes 9 values of n.

   For $f(n)=2$, which occurs for $n\in[10,99]$, $g(n)=2$ works, which occurs for $n\in[4,15]$. The intersection is $n\in[10,15]$. Here $g(n)=4$ also works, which occurs when $n\in[64,255]$ The intersection yields $n\in[64,99]$. Any larger values of $g(n)$ will require $n>99$, so we have $6+36=42$ values of n in this case.

   For $f(n)=3$, which occurs for $n\in[100,999]$, $g(n)$ takes the values of 4 or 5 neither of which are multiples of 3.

   Now, for $n\geq 1000$, consider $X=\frac{g(n)}{f(n)}$. When $n=1000$, $X=\frac{5}{4}$. For each integer $i\geq 2$ consider what happens when n increases from $n=10^i$ to $n=10^{i+1}$. Since $4<10<4^2$, each interval $[10^i,10^{i+1})$ is guaranteed to contain one or two powers of 4. Thus, when n increases from $n=10^i$ to $n=10^{i+1}$, the numerator increases by 1 or 2 before n hits 10^{i+1} at which point the denominator increases by 1. Thus, for $n\geq 1000$ the value of X will forever remain strictly between 1 and 2, meaning that $g(n)$ will never be a multiple of $f(n)$.

   The final answer is $9+42=\boxed{51}$.

   Proposed by Jerry Tan

Round 5

1. **Problem:** Julia baked a pie for herself to celebrate pi day this year. If Julia bakes anyone pie on pi day, the following year on pi day she bakes a pie for herself with $1/3$ probability, she bakes her friend a pie with $1/6$ probability, and she doesn't bake anyone a pie with $1/2$ probability. However, if Julia doesn't make pie on pi day, the following year on pi day she bakes a pie for herself with $1/2$ probability, she bakes her friend a pie with $1/3$ probability, and she doesn't bake anyone a pie with

1/6 probability. The probability that Julia bakes at least 2 pies on pi day in the next 5 years can be expressed as $\frac{p}{q}$, for relatively prime positive integers p and q. Compute $p+q$.

Solution: We only care about whether or not she bakes a pie at all, so we make the simplification that if she bakes pie one year, there is a 1/2 chance she bakes one the following year, otherwise there is a 5/6 chance she bakes one the following year. We use P to denote she baked a pie N to denote otherwise, and we use a string of five such letters to denote the possibilities over the five years. We use complementary counting.

Case 1: 0 pies. This occurs with probability $\frac{1}{2}\left(\frac{1}{6}\right)^4 = \frac{1}{2^5 3^4}$

Case 2: 1 pie.

Subcase 2.1: PNNNN. The probability is $\frac{1}{2} \cdot \frac{1}{2}\left(\frac{1}{6}\right)^3 = \frac{1}{2^5 3^3}$.

Subcase 2.2: NPNNN, NNPNN, or NNNPN. The combined probability is $3 \cdot \frac{1}{2} \cdot \frac{5}{6} \cdot \frac{1}{2}\left(\frac{1}{6}\right)^2 = \frac{5}{2^5 3^2}$.

Subcase 2.3: NNNNP. The probability is $\frac{1}{2}\left(\frac{1}{6}\right)^3 \cdot \frac{5}{6} = \frac{5}{2^5 3^4}$.

Adding it up, we get $\frac{1+3+45+5}{2^5 3^4} = \frac{54}{2^5 3^4} = \frac{1}{48}$. The desired probability is then $\frac{47}{48}$ for answer of $\boxed{95}$.

Proposed by Annie Wang

2. **Problem:** Steven is flipping a coin but doesn't want to appear too lucky. If he flips the coin 8 times, the probability he only gets sequences of consecutive heads or consecutive tails that are of length 4 or less can be expressed as $\frac{p}{q}$, for relatively prime positive integers p and q. Compute $p+q$.

   Solution: We will proceed by complementary counting, and consider the cases of sequences of length $5, 6, 7$, and 8.
   Case 1: 8 in a row
   There are 2 possibilities since we either choose all heads or all tails.
   Case 2: 7 in a row
   We consider only the sequences starting with 7 heads. There must be one of these since the last coin must be a tail. Then we multiply by 4 to get 4 total possibilities to account for reflecting the sequence and swapping heads and tails.
   Case 3: 6 in a row
   Again, only consider the sequences starting with 6 heads. The seventh coin must be a tail, and there are two choices for the eighth. This gives $2 \cdot 2 \cdot 2 = 8$ considering symmetry. Now we consider the case where the sequence of six heads begins with the second coin. There is only one possibility since the first and eighth coin must both be tails, giving 2 possibilities considering flipping heads to tails. The total for this case is 10.
   Case 4: 5 in a row
   Using the same strategy, if we start with 5 heads, then the sixth coin must be a tail, and we have $2 \cdot 2 = 4$ options for the remaining coins. Multiplying by 4 to account for symmetries gives 16 possibilities. If the sequence of 5 heads begins with the second coin, the first and the seventh coin must be a tail, giving 2 options for the eighth. Multiply by 4 to get 8 possibilities. In total we have 24 possibilities for this case.
   Overall we have $2^8 = 256$ sequences, and 40 sequences with consecutive subsequences of length 5 or greater. Thus the probability is $\frac{256-40}{256} = \frac{216}{256} = \frac{27}{32}$. Therefore the answer is $27 + 32 = \boxed{59}$.

   Proposed by Devin Brown

3. **Problem:** Let $ABCD$ be a square with side length 3. Further, let E be a point on side AD, such that $AE = 2$ and $DE = 1$, and let F be the point on side AB such that triangle CEF is right with hypotenuse CF. The value CF^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

   Solution: First let $AF = x$, and $BF = 3 - x$. By the pythagorean theorem, we have $CE^2 + EF^2 = CF^2$, as well as $CE^2 = 3^2 + 1^2 = 10$, $EF^2 = x^2 + 2^2 = x^2 + 4$, and $CF^2 = (3-x)^2 + 3^2 = x^2 - 6x + 18$. Combining these, we have $x^2 + 14 = x^2 - 6x + 18$, which simplifies to give us $x = \frac{2}{3}$. Now we plug in to find $CF^2 = (3 - \frac{2}{3})^2 + 9 = \frac{49}{9} + 9 = \frac{130}{9}$. Therefore the answer is $130 + 9 = \boxed{139}$.

   Proposed by Devin Brown

Round 6

1. **Problem:** Let P be a point outside circle ω with center O. Let A, B be points on circle ω such that PB is a tangent to ω and $PA = AB$. Let M be the midpoint of AB. Given $OM = 1, PB = 3$, the value of AB^2 can be expressed as $\dfrac{m}{n}$ for relatively prime positive integers m, n. Find $m+n$.

   Solution:

   Note $\angle APB = \angle ABP = \dfrac{1}{2}\angle AOB = \angle BOM$ where $\angle APB = \angle ABP$ becomes from the fact that $\triangle APB$ is isoceles and $\angle ABP = \angle AOM$ comes from the fact that PB is tangent to ω. Note that $\angle OMA = \angle OMB = 90°$ since M is a midpoint.

   Let the altitude from A onto PB be Y. Note that Y is the midpoint of BP as $\triangle APB$ is isosceles. Additionally, $PY = YB = \dfrac{3}{2}$. Let $PA = AB = x$. We want to find the value of x. Note that by Pythagoras, we have $OB = \sqrt{1^2 + \left(\dfrac{x}{2}\right)^2} = \sqrt{1 + \dfrac{x^2}{4}}$. Since $\angle ABY = \angle BOM$ we have $\triangle ABY \sim \triangle BOM$. Thus, we have $\dfrac{AB}{BY} = \dfrac{OB}{OM} \Rightarrow AB \cdot OM = BY \cdot OB \Rightarrow x = \dfrac{3}{2} \cdot \sqrt{1 + \dfrac{x^2}{4}}$. Solving yields $x = \dfrac{6\sqrt{7}}{7} \Rightarrow x^2 = \dfrac{36}{7} \Rightarrow m + n = \boxed{43}$.

   Proposed by Jerry Li

2. **Problem:** Let $a_0, a_1, a_2...$ with each term defined as $a_n = 3a_{n-1} + 5a_{n-2}$ and $a_0 = 0, a_1 = 1$. Find the remainder when a_{2020} is divided by 360.

   Solution: We can find $a_{2020} \pmod{5, 8, 9}$ and then use Chinese Remainder Theorem to find the remainder when divided by 360. To find each of the parts, we simply list out terms until we find a pattern that continues for 2 terms.

   Part 1: $\pmod 5$. The sequence $\pmod 5$ goes: $0, 1, 3, 4, 2, 1, 3, 4, \cdots$ The sequence repeats every 4 terms. Since $2020 \equiv 0 \pmod 4$ we know that $a_{2020} \pmod 5$ equals a_4, which is 2.

   Part 2: $\pmod 8$. The sequence $\pmod 8$ goes: $0, 1, 3, 6, 1, 1, 0, 5, 7, 6, 5, 5, 0, 1, 3, \cdots$ The sequence repeats every 12 terms. Note that $2020 \equiv 4 \pmod{12}$ so $a_{2020} \pmod 8$ is a_4, which is 1.

   Part 3: $\pmod 9$. The sequence $\pmod 9$ goes: $0, 1, 3, 5, 3, 7, 0, 8, 6, 4, 6, 2, 0, 1, 3, \cdots$ The sequence repeats every 12 terms. Since $2020 \equiv 4 \pmod{12}$ we are looking for a_4, which is 3.

   Thus, we know that

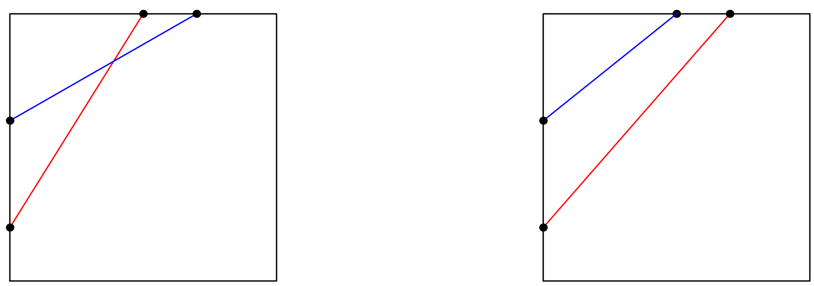
$$a_{2020} \equiv 2 \pmod 5$$
$$a_{2020} \equiv 1 \pmod 8$$
$$a_{2020} \equiv 3 \pmod 9$$

Using Chinese Remainder theorem yields $a_{2020} \equiv \boxed{57} \pmod{360}$.

Proposed by Jerry Li

3. **Problem:** James and Charles each randomly pick two points on distinct sides of a square, and they each connect their chosen pair of points with a line segment. The probability that the two line segments intersect can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

Solution: We will consider the four points chosen in total as indistinguishable. There are five possible distinct distributions of points among the four sides, namely $1-1-1-1$; $1-2-1-0$; $2-1-1-0$; $2-2-0-0$; $2-0-2-0$. In all the distributions except the first one, there are two equally probable ways for the two segments to have been selected to create that distribution, and one way results in an intersection while the other does not. An example is shown for $2-2-0-0$.



Thus in this case, the probability of intersection is $\frac{1}{2}$.

Now we focus on the $1-1-1-1$ case. WLOG the point on the left side is from James. There are three equally probable ways that two segments created this $1-1-1-1$ distribution, and exactly one choice of segments results in an intersection. With James' choice of two points fixed, Charles always has $\frac{1}{\binom{4}{2}} = \frac{1}{6}$ chance of choosing his points to form the $1-1-1-1$ distribution. Thus this case has a $\frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18}$ chance of an intersection. Then, the previous case has $\left(1 - \frac{1}{6}\right) \cdot \frac{1}{2} = \frac{5}{12}$ chance of an intersection. The total probability is $\frac{1}{18} + \frac{5}{12} = \frac{17}{36}$ for an answer of $\boxed{53}$.

Proposed by Jerry Tan

Round 7

1. **Problem:** For some positive integers x, y let $g = \gcd(x, y)$ and $\ell = \operatorname{lcm}(2x, y)$. Given that the equation $xy + 3g + 7\ell = 168$ holds, find the largest possible value of $2x + y$.

Solution: Denote $v_2(x)$ the exponent of the largest power of 2 dividing x. We have two cases:

Case 1: $v_2(y) > v_2(x)$. Then $\ell = \operatorname{lcm}(2x, y) = \operatorname{lcm}(x, y)$. We use the identity $xy = \gcd(x, y) \cdot \operatorname{lcm}(x, y)$ and rearrange to get

$$(g+7)(\ell+3) = 189 = 3^3 \cdot 7.$$

We know g, ℓ are positive integers and g divides ℓ. The possible pairs for $((g+7), (\ell+3))$ are then $\{(9, 21), (21, 9), (27, 7)\}$ which correspond to $(g, \ell) \in \{(2, 18), (14, 6), (20, 4)\}$, of which only $(2, 18)$ satisfies g divides ℓ. Then we find $(x, y) = (2, 18), (18, 2)$, both of which fail $v_2(y) > v_2(x)$. Thus this case has no solution.

Case 2: $v_2(y) \le v_2(x)$. Then $\ell = \operatorname{lcm}(2x, y) = 2\operatorname{lcm}(x, y)$. Then $xy = \frac{g\ell}{2}$, and substituting we find

$$(g+14)(\ell+6) = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7.$$

From here we proceed similarly as in Case 1, and we eventually find $(g\ell) \in \{(1,22),(6,15),(7,14)\}$ are the only possibilities with $g, \ell > 0$ and $g \leq \ell$. We see that $(1,22)$ and $(7,14)$ satisfy $g|\ell$. Remembering that $\ell = 2\,\text{lcm}(x,y)$, these correspond to $(x,y) = (1,11),(11,1),(7,7)$, all of which satisfy $v_2(y) \leq v_2(x)$. Then the largest $2x+y$ is $\boxed{23}$.

Proposed by Jerry Tan

2. **Problem:** Marco writes the polynomials $f(x) = nx^4 + 2x^3 + 3x^2 + 4x + 5$ and $g(x) = a(x-1)^4 + b(x-1)^3 + 6(x-1)^2 + d(x-1) + e$, where n, a, b, d, e are real numbers. He notices that $g(i) = f(i) - |i|$ for each integer i satisfying $-5 \leq i \leq -1$. Then n^2 can be expressed as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p+q$.

Solution: Note that the polynomial $g(x) - f(x) - x$ has degree at most four but has five roots, namely the integers $-5 \leq x \leq -1$. Thus $g(x) - f(x) - x$ must be the zero polynomial, so $g(x) - f(x) - x = 0$ for all reals x. Plugging in $x = 0, 1$ and 2, and simplifying we find

$$a - b + 6 - d + e - 5 = 0 \Rightarrow\ a - b - d + e = -1$$
$$e - (n+14) - 1 = 0 \Rightarrow\ e = n + 15$$
$$a + b + 6 + d + e - (16n + 41) - 2 = 0 \Rightarrow\ a + b + d + e = 16n + 37,$$

respectively. We add the first and third equations to obtain $2a + 2e = 16n + 36 \ \Rightarrow\ a + e = 8n + 18$. Substituting the second equation we get $a + n + 15 = 8n + 18$. Finally, comparing x^4 coefficients of f and g we see $n = a$, so

$$2n + 15 = 8n + 18 \Rightarrow\ n = -\frac{1}{2}.$$

Then $n^2 = \frac{1}{4}$ so the answer is $\boxed{5}$.

Proposed by Jerry Tan

3. **Problem:** Equilateral $\triangle ABC$ is inscribed in a circle with center O. Points D and E are chosen on minor arcs AB and BC, respectively. Segment $\overline{CD}$ intersects $\overline{AB}$ and $\overline{AE}$ at Y and X, respectively. Given that $\triangle DXE$ and $\triangle AXC$ have equal area, $\triangle AXY$ has area 1, and $\triangle ABC$ has area 52, find the area of $\triangle BXC$.

Solution: First, $\triangle AXC \sim \triangle DXE$ by AA similarity, so if they have the same area, then they are congruent. Then $DE = AC$ and since equal chords subtend equal arcs, minor arc DE is $120°$. This implies minor arcs AD and BE have equal measure, so $m\angle DCA = m\angle EAB$. Let $\overline{AE}$ intersect $\overline{BC}$ at Z. Then $\triangle BAZ \cong \triangle ACY$ so

$$[BZXY] = [AXC].$$

If we let $[AXC] = a$, then $[ZXC] = 51 - 2a$. Now,

$$a + 1 = \frac{[BAZ]}{[XAY]} = \frac{BA}{YA} \cdot \frac{ZA}{XA} = \frac{[CAB]}{[CAY]} \cdot \frac{[CAZ]}{[CAX]} = \frac{52}{a+1} \cdot \frac{51-a}{a}.$$

This is a cubic in a, and we know there is exactly one positive real solution since the configuration is determined by the conditions. We can hope for a rational root, and by Rational Root Theorem any rational roots are integers dividing $52 \cdot 51$. It is not hard to see that $a = 12$ works (keep the equation in the above fraction form for easy cancellation). By dropping altitudes from X and A to $\overline{BC}$, we get

$$\frac{[BXC]}{[BAC]} = \frac{ZX}{ZA} = \frac{[ZXC]}{[ZAC]} = \frac{27}{39},$$

so $[BXC] = \boxed{36}$.

Proposed by Jerry Tan

Round 8

Problem: Let A be the number of total webpage visits our website received last month. Let B be the number photos in our photo collection from ABMC onsite 2017. Let M be the mean speed round score. Further, let C be the number of times the letter c appears in our problem bank. Estimate

$$A \cdot B + M \cdot C.$$

Your answer will be scored according to the following formula, where X is the correct answer and I is your input.

$$\max\left\{0, \left\lceil \min\left\{13 - \frac{|I-X|}{0.05|I|}, 13 - \frac{|I-X|}{0.05|I-2X|}\right\}\right\rceil\right\}.$$

Solution: 3148 views in March, 115 photos, 9.87 mean score on speed round, and 1157 c's in our problem bank. $AB + MC = \boxed{373439.59}$

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