Acton-Boxborough Math Competition Online Contest Solutions

Friday, December 20 — Sunday, December 22, 2019

1. **Problem:** Let a be an integer. How many fractions $\frac{a}{100}$ are greater than $\frac{1}{7}$ and less than $\frac{1}{3}$?.

   Solution: We know that $\frac{1}{7} = 0.1428...$ and $\frac{1}{3} = 0.3333...$, so clearly, a can only be any integer from 15 to 33. There are thus $\boxed{19}$ such fractions.

   Proposed by Aaron Zhang

2. **Problem:** Justin Bieber invited Justin Timberlake and Justin Shan to eat sushi. There were 5 different kinds of fish, 3 different rice colors, and 11 different sauces. Justin Shan insisted on a spicy sauce. If the probability of a sushi combination that pleased Justin Shan is 6/11, then how many non-spicy sauces were there?

   Solution: The choice of fish or rice is irrelevant in this problem. Since the probability of a combination that Justin approves of is $\frac{6}{11}$, it must be that 6 of the 11 sauces are spicy sauces. Thus, there are $\boxed{5}$ non-spicy sauces.

   Proposed by Eddie Wang

3. **Problem:** A palindrome is any number that reads the same forward and backward (for example, 99 and 50505 are palindromes but 2020 is not). Find the sum of all three-digit palindromes whose tens digit is 5.

   Solution: The palindromes are $151, 252, 353, \ldots, 959$. The sum of these numbers is $\boxed{4995}$

   Proposed by Aaron Zhang

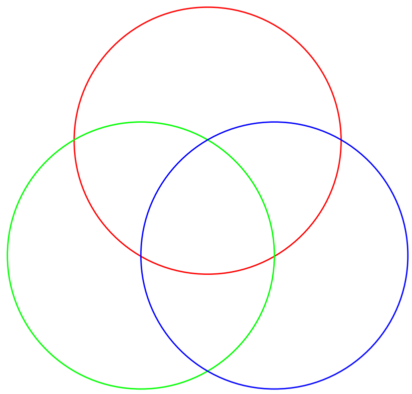
4. **Problem:** Isaac is given an online quiz for his chemistry class in which he gets multiple tries. The quiz has 64 multiple choice questions with 4 choices each. For each of his previous attempts, the computer displays Isaac's answer to that question and whether it was correct or not. Given that Isaac is too lazy to actually read the questions, the maximum number of times he needs to attempt the quiz to guarantee a 100% can be expressed as 2^{2^k}. Find k.

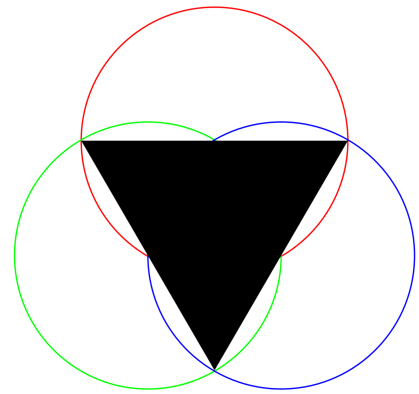
   Solution: Isaac can in fact get all the questions right in only 4 attempts. First, he answers choice A for all the questions. Then on his second attempt, for all the questions he gets wrong, he answers B. On the third attempt he answers C for all the wrong ones, then D on his fourth attempt, his guarantees that he gets 100%. Since $4 = 2^{2^1}$, the answer is $\boxed{1}$.

   Proposed by Jerry Tan

5. **Problem:** Consider a three-way Venn Diagram composed of three circles of radius 1. The area of the entire Venn Diagram is of the form $\frac{a}{b}\pi + \sqrt{c}$ for positive integers a, b, c where a, b are relatively prime. Find $a + b + c$. (Each of the circles passes through the center of the other two circles)

   Solution: Let the three points that each pair of circles intersect at, not including the centers of the circles, be points A, B, C. By symmetry ABC is an equilateral triangle passing through the centers of the cricles. Thus, the venn diagram can be decomposed into an equilateral triangle of sides length 2 and 3 semicircles of raidus 1. Thus the area is $\frac{3}{2}\pi + \sqrt{3}$, so the answer is $\boxed{8}$.





Proposed by Poonam Sahoo

6. **Problem:** The sum of two four-digit numbers is 11044. None of the digits are repeated and none of the digits are 0s. Eight of the digits from 1-9 are represented in these two numbers. Which one is not?

   Solution: To begin, the one's digit of either number clearly cannot be 2, 4 or 7; it also can't be 1 or 3 because if the sum of the ones digit doesn't carry over, then there would be no 2 numbers remaining that sum to 4 again. Thus, the ones digits must either be 5 and 9 or 6 and 8. If we use 5 and 9 as our ones digits, then the sum of the tens digits must have a ones digit of 3. Our only two choices for the tens digits are either 1 and 2 or 6 and 7. Choosing 1 and 2 for the tens digits means we will need 2 pairs of digits that each sum to ten for the hundreds and thousands digits. Since we have the pairs of digits 3,7 and 4,6 left, this works out perfectly. The only digit that is not used is thus $\boxed{8}$.

   Proposed by Justin Shan

7. **Problem:** Al wants to buy cookies. He can buy cookies in packs of 13, 15, or 17. What is the maximum number of cookies he can not buy if he must buy a whole number of packs of each size?

   Solution: Consider the number of cookies mod 13. If we can make every residue mod 13, we can just add multiples of 13. Note that 15 and 17 are $2, 4$ mod 13. Thus, all the even residues mod 13 are relatively "easy" to reach, for example to get 10 mod 13 cookies simply take $17 + 17 + 15$. Out of the odd residues, 11 requires the most summands, $6 * 17 = 102$. We can clearly make all numbers of cookies 11 mod 13 greater than or equal to 102. However, this means $102 - 13 = 89$ is impossible to make, while the other residues can all be achieved with fewer cookies, thus $\boxed{89}$ is the answer.

   Proposed by

8. **Problem:** Let $\triangle ABC$ be a right triangle with base $AB = 2$ and hypotenuse $AC = 4$ and let AD be a median of $\triangle ABC$. Now, let BE be an altitude in $\triangle ABD$ and let DF be an altitude in $\triangle ADC$. The quantity $(BE)^2 - (DF)^2$ can be expressed as a common fraction $\frac{a}{b}$ in lowest terms. Find $a + b$.

   Solution: Begin by noticing that $\triangle ABC$ must be a $30 - 60 - 90$ triangle with a 30° angle at $\angle C$ which implies that $BC = 2\sqrt{3}$. Since AD is a median, D is the midpoint of BC which implies that $BD = DC = \sqrt{3}$.

   Let's first find BE. We know that $\triangle ABD$ is a right triangle with bases $AB = 2$ and $BD = \sqrt{3}$. By the Pythagorean Theorem, $AD = \sqrt{7}$. Furthermore, we know that $AB \times BD = BE \times AD$ (both are equal to twice the area of $\triangle ABD$). Thus, $BE = \frac{2\sqrt{3}}{\sqrt{7}}$.

Now, let's find DF. If we let $AF = x$, then $CF = 4 - x$. We can then make a system of equations (using the Pythagorean Theorem) to solve for DF. We have that $x^2 + DF^2 = 7$ and that $(4-x)^2 + DF^2 = 3$. After subtracting the second equation from the first, we get $8x - 16 = 4$. Therefore, $x = \frac{5}{2}$ which implies $DF = \frac{\sqrt{3}}{2}$.

Our desired quantity is $(BE)^2 - (DF)^2 = \frac{12}{7} - \frac{3}{4} = \frac{27}{28}$. The answer is $27 + 28 = \boxed{55}$.

Proposed by Aaron Zhang

9. **Problem:** Let $P(x)$ be a monic cubic polynomial with roots r, s, t, where t is real. Suppose that $r + s + 2t = 8, 2rs + rt + st = 12$ and $rst = 9$. Find $P(2)$.

   Solution: We will first try to solve for t, since we have 3 equations and 3 variables. Number the equations (i), (ii), (iii) in the same order as in the problem statement. We can substitute $rs = \dfrac{9}{t}$ from (iii) and $r + s = 8 - 2t$ from (i) into (ii) to get $12 = 2rs + rt + st = \dfrac{18}{t} + t(8 - 2t)$ which simplifies to $t^3 - 4t^2 + 6t - 9 = 0$. We use RRT and see $t = 3$ is a root so we factor $(t-3)(t^2 - t + 3)$. The quadratic has nonreal roots so $t = 3$, so plugging back in we get $r + s = 2$ and $rs = 3$. Thus by vieta's r, s satisfy $x^2 - 2x + 3 = 0$. Then accounting for $(t-3) = 0$ we know $P(x) = (x^2 - 2x + 3)(x - 3)$, so $|P(2)| = |(3)(-1)| = \boxed{3}$.

   Proposed by Nithin Kavi

10. **Problem:** Let S be the set $\{1, 2, ..., 21\}$. How many 11-element subsets T of S are there such that there does not exist two distinct elements of T such that one divides the other?

    Solution: Note that every positive integer can be written as $2^k \cdot m$ for some nonnegative integer k and odd integer m. We put the numbers 1 to 21 into 11 groups, $g_1, g_3, g_5, ..., g_{21}$ such that the elements in g_i are of the form $2^k \cdot i$. For example $g_5 = \{5, 10, 20\}$. By the problem condition we can have at most one element from each group in T, so we must have exactly one elements from each group. Thus, we must have $11, 13, 15, 17, 19, 21$ in T. Since $7|21$ and $1, 3, 5|15$ we can't choose $1, 3, 5$, or 7 to be in T, and we must choose 14 from g_7. We also can't choose 2, otherwise we wouldn't have any elements from g_5. Now we have 3 cases based on what we choose from g_1.

    Case 1: 4 is in T. Then we can't have 12 or 20 in T, so we must have 6 and 10, meaning we can't take 18 so we must take 9. This gives 1 subset T.

    Case 2: 8 is in T. We have 2 options from each of g_3, g_5, g_9, and as long as we don't pick both 6 and 18 we are fine. Thus there are $2^3 - 2 = 6$ subsets T.

    Case 3: 16 is in T. Identical to case 2, we have $2^3 - 2 = 6$ subsets.

    Thus in total there are $\boxed{13}$ options for T.

    Proposed by Jerry Tan