Acton-Boxborough Math Competition 2019 Solutions

ABMC Team

April 26, 2019

Speed Round

1. **Problem:** What is $2019 + 201 + 20 + 2$?

   Solution: We have
   $$2019 + 201 + 20 + 2 = \boxed{2242}.$$

   Proposed by Allen Wang

2. **Problem:** The sequence $100, 102, 104, \ldots, 996$, and 998 is the sequence of all three-digit even numbers. How many three digit even numbers are there?

   Solution: There are 900 three digit integers ranging from 100 to 999. Exactly half of them are even, so there are $900/2 = \boxed{450}$ even three-digit integers.

   Proposed by Allen Wang

3. **Problem:** Find the units digit of $25 \times 37 \times 113 \times 22$.

   Solution: Notice that the product of the units digit of the first and last numbers is 0. Multiplying any number with a units digit of 0 by any other number will produce a final units digit of $\boxed{0}$.

   Proposed by Eddie Wang

4. **Problem:** Samuel has a number in his head. He adds 4 to the number and then divides the result by 2. After doing this, he ends up with the same number he had originally. What is his original number?

   Solution: Call Samuel's number x. We can model his situation as
   $$\frac{x+4}{2} = x$$

   Solving this equation, we get that $x = \boxed{4}$

   Proposed by Charlotte Wang

5. **Problem:** According to Shay's Magazine, every third president is terrible (so the third, sixth, ninth president and so on were all terrible presidents). If there have been 44 presidents, how many terrible presidents have there been in total?

   Solution: We notice that every multiple of 3 is a terrible president. The greatest multiple of 3 that is less than or equal to 44 is 42, which is 3×14. Therefore, there have been $\boxed{14}$ terrible presidents.

   Proposed by Eddie Wang

6. **Problem:** In the game Tic-Tac-Toe, a player wins by getting three of his or her pieces in the same row, column, or diagonal of a 3×3 square. How many configurations of 3 pieces are winning? Rotations and reflections are considered distinct.

   Solution: There are 3 cases: 3 in a row, column, or diagonal. There are 3 rows, 3 columns, and 2 diagonals. Thus the answer is $3 + 3 + 2 = \boxed{8}$.

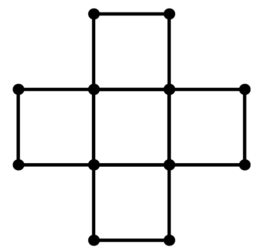
   Proposed by Allen Wang

7. **Problem:** Eddie is a sad man. Eddie is cursed to break his arm 4 times every 20 years. How many times would he break his arm by the time he reaches age 100?

   Solution: We say that he breaks his arm once every five years. Then since $\frac{100}{5} = 20$ he will break his arm in $\boxed{20}$ places.

   Proposed by Eddie Wang

8. **Problem:** The figure below is made from 5 congruent squares. If the figure has perimeter 24, what is its area?



Solution: The perimeter consists of 12 equal sides. So, each side is $24/12 = 2$. Then the area is $5 \cdot 2^2 = \boxed{20}$

Proposed by Akshay Gowrishankar

9. **Problem:** Sancho Panza loves eating nachos. If he eats 3 nachos during the first minute, 4 nachos during the second, 5 nachos during the third, how many nachos will he have eaten in total after 15 minutes?

Solution: Sancho Panza will eat 17 nachos on his last day, so this problem can be modeled by the expression $3 + 4 + \cdots + 17$. This is an arithmetic progression, so the sum is $\frac{3+17}{2} \cdot (17 - 3 + 1) = 10 \cdot 15 = \boxed{150}$

Proposed by Eddie Wang

10. **Problem:** If the day after the day after the day before Wednesday was two days ago, then what day will it be tomorrow?

Solution: Two days ago it was Thursday, so today it is Saturday. Therefore, tomorrow will be $\boxed{\text{Sunday}}$.

Proposed by Eddie Wang

11. **Problem:** Neetin the Rabbit and Poonam the Meerkat are in a race. Poonam can run at 10 miles per hour, while Neetin can only hop at 2 miles per hour. If Neetin starts the race 2 miles ahead of Poonam, how many minutes will it take for Poonam to catch up with him?

Solution: Notice that Poonam runs at a rate 8 miles per hour faster than Neetin. Nithin has a 2 mile lead on her, so it will take her

$$\frac{1 \text{ hour}}{8 \text{ miles}} \times 2 \text{ miles}$$

$= \frac{2}{8}$ hours = $\boxed{15}$ minutes.

Proposed by Akshay Gowrishankar

12. **Problem:** Dylan has a closet with t-shirts: 3 gray, 4 blue, 2 orange, 7 pink, and 2 black. Dylan picks one shirt at random from his closet. What is the probability that Dylan picks a pink or a gray t-shirt?

Solution: Dylan has 3 gray shirts and 7 pinks shirts, or 10 pink or gray shirts. He has $3 + 4 + 2 + 7 + 2$, or 18 total shirts. Thus, the probability of picking a pink or grey shirt is $10/18 = \boxed{\frac{5}{9}}$.

Proposed by Eddie Wang

13. **Problem:** Serena's brain is 200% the size of Eric's brain, and Eric's brain is 200% the size of Carlson's. The size of Carlson's brain is what percent the size of Serena's?

    Solution: We have that Serena's brain is twice the size of Eric's, which is twice the size Carlson's brain, so Serena's brain is 4 times Carlson's. Then $\frac{1}{4} = \boxed{25\%}$.

    Proposed by Akshay Gowrishankar

14. **Problem:** Find the sum of the coefficients of $(2x+1)^3$ when it is fully expanded.

    Solution: Expanding $(2x+1)^3$, we get that it is equal to $8x^3 + 12x^2 + 6x + 1$. The sum of the coefficients is therefore $8 + 12 + 6 + 1 = \boxed{27}$

    Proposed by Antonio Frigo

15. **Problem:** Antonio loves to cook. However, his pans are weird. Specifically, the pans are rectangular prisms without a top. What is the surface area of the outside of one of Antonio's pans if their volume is 210, and their length and width are 6 and 5, respectively?

    Solution: Since the volume is 210 and the length of two of the sides are 5 and 6, the length of the height must be $\frac{210}{5 \cdot 6} = 7$. The total surface area is therefore $2 \cdot (5 \times 6 + 5 \times 7 + 6 \times 7)$; however, we need to subtract the surface area of the top which is 5×6. Therefore, the answer is $2 \cdot (30 + 35 + 42) - 30 = \boxed{184}$

    Proposed by Eddie Wang

16. **Problem:** A lattice point is a point on the coordinate plane with 2 integer coordinates. For example, (3, 4) is a lattice point since 3 and 4 are both integers, but (1.5, 2) is not since 1.5 is not an integer. How many lattice points are on the graph of the equation $x^2 + y^2 = 625$?

    Solution: Notice that 625 is 25^2. Begin by looking at possible Pythagorean triples where the greatest number is a factor of 25. The possible triples are $\{3, 4, 5\}$ and $\{7, 24, 25\}$. Therefore, $\{x, y\}$ can be

    $$\{\pm 15, \pm 20\}, \{\pm 20, \pm 15\}, \{\pm 7, \pm 24\}, \{\pm 24, \pm 7\}.$$

    There are 16 possibilities this way. Notice that $\{x, y\}$ can also be $\{0, \pm 25\}$ or $\{\pm 25, 0\}$. This gives 4 other solutions. Therefore there are a total of $\boxed{20}$ lattice points.

    Proposed by Aaron Zhang

17. **Problem:** Jonny has a beaker containing 60 liters of 50% saltwater (50% salt and 50% water). Jonny then spills the beaker and 45 liters pour out. If Jonny adds 45 liters of pure water back into the beaker, what percent of the new mixture is salt?

    Solution: After 45 liters are poured out, there are 15 liters left of the original mixture, and therefore there are 7.5 liters of salt in the 15 liters. After 45 liters of pure water are added, there will be 7.5 liters of salt out of 60 total liters of mixture. This equates to $\boxed{12.5}$ percent of salt.

    Proposed by Akshay Gowrishankar

18. **Problem:** There are exactly 25 prime numbers in the set of positive integers between 1 and 100, inclusive. If two not necessarily distinct integers are randomly chosen from the set of positive integers from 1 to 100, inclusive, what is the probability that at least one of them is prime?

    Solution: The probability that at least one is prime is $1-$ the probability that neither is prime. There are $100 - 25 = 75$ non-prime numbers out of 100 total, thus the probability that the number is not prime is $75/100 = 3/4$. For both numbers not to be prime would be $(\frac{3}{4})^2 = \frac{9}{16}$. Finally, $1 - \frac{9}{16} = \boxed{\frac{7}{16}}$

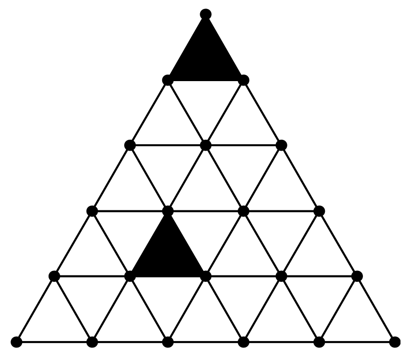
    Proposed by Aaron Zhang

19. **Problem:** How many consecutive zeroes are at the end of 12! when it is expressed in base 6?

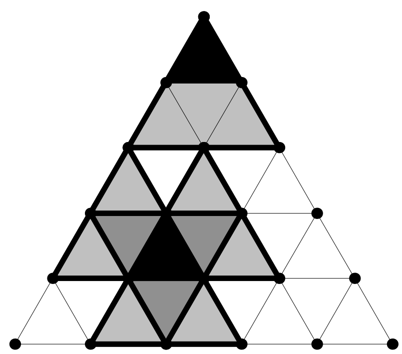
    Solution: We want to find the greatest power of 6 which divides 12!. This is equivalent to the greatest power of 3 that divides 12! which is 3^5. Therefore, the greatest power of 6 that divides 12! is 6^5, and thus, there are $\boxed{5}$ consecutive zeroes at the end when 12! is expressed in base 6.

    Proposed by Nithin Kavi

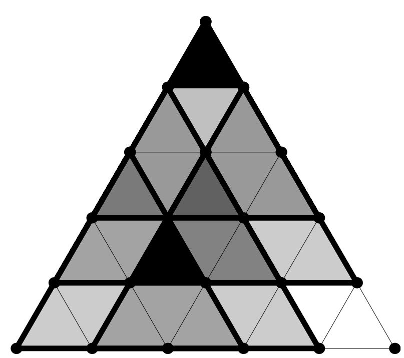
20. **Problem:** Consider the following figure. How many triangles with vertices and edges from the following figure contain exactly 1 black triangle?



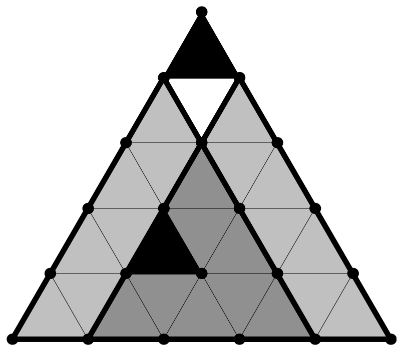
Solution: For triangles of slide length 1, we have two cases, namely the black triangles themselves.



For triangles of side length 2, we consider each black triangle in turn. For the top, there is exactly one case. For the bottom, there are four. So, there are five in total for side length 2.



For triangles of side length 3, for the top, once again, there is exactly one case. For the bottom, there are four. Hence, there are five in total for side length 3.



For triangles of side length 4, for the top, there are none, since they would contain the bottom black triangle. For the bottom, there are two. Thus, for side length 4 there exist 2 triangles.

Adding these all together, we have $2+5+5+2=\boxed{14}$

Proposed by Antonio Frigo

21. **Problem:** After Akshay got kicked off the school bus for rowdy behavior, he worked out a way to get home from school with his dad. School ends at 2:18 pm, but since Akshay walks slowly he doesn't get to the front door until 2:30. His dad doesn't like to waste time, so he leaves home everyday such that he reaches the high school at exactly 2:30 pm, instantly picks up Akshay and turns around, then drives home. They usually get home at 3:30 pm. However, one day Akshay left school early at exactly 2:00 pm because he was expelled. Trying to delay telling his dad for as long as possible, Akshay starts jogging home. His dad left home at the regular time, saw Akshay on the way, picked him up and turned around instantly. They then drove home while Akshay's dad yelled at him for being a disgrace. They reached home at 3:10 pm. How long had Akshay been walking before his dad picked him up?

    Solution: Since Akshay's dad usually reaches home at $3:30$, it takes him an hour to reach the school. That means he starts out at $1:30$. Since he must spend an equal amount of time driving both ways and there are 100 minutes spent so therefore he reached Akshay at 50 minutes past $1:30$, i.e. $2:20$. Since Akshay left at $2:00$, he had been walking for $\boxed{20}$ minutes before getting picked up.

    Proposed by Nithin Kavi

22. **Problem:** In quadrilateral $ABCD$, diagonals AC and BD intersect at O. Then $\angle BOC = \angle BCD$, $\angle COD = \angle BAD$, $AB = 4$, $DC = 6$, and $BD = 5$. What is the length of BO?

    Solution: Let $BO = x$. Then $DO = 5 - x$. Since

    $$180 = \angle COD + \angle BOC = \angle BAD + \angle BCD,$$

    we have that $ABCD$ is a cyclic quadrilateral. Then by equal inscribed angles we have $\angle BAC = \angle BDC$, and $\angle ABD = \angle ACD$, so

    $$\begin{aligned} \triangle BAO \sim \triangle CDO \Rightarrow\ & \frac{DC}{AB} = \frac{CO}{BO} \\ \Rightarrow\ & \frac{6}{4} = \frac{CO}{x} \\ \Rightarrow\ & CO = \frac{3}{2}x. \end{aligned}$$

    Since $\angle BOC = \angle BCD$, we also have

    $$\triangle BOC \sim \triangle BCD \Rightarrow \frac{BO}{BC} = \frac{BC}{BD} = \frac{CO}{CD}$$

$$\Rightarrow \frac{x}{BC} = \frac{BC}{5} = \frac{\frac{3}{2}x}{6}$$

$$\Rightarrow BC = \sqrt{5x}$$

$$\Rightarrow \frac{x}{\sqrt{5x}} = \frac{x}{4} \Rightarrow 5x = 16 \Rightarrow x = \boxed{\frac{16}{5}}.$$

Proposed by Poonam Sahoo

23. **Problem:** A standard six-sided die is rolled. The number that comes up first determines the number of additional times the die will be rolled (so if the first number is 3, then the die will be rolled 3 more times). Each time the die is rolled, its value is recorded. What is the expected value of the sum of all the rolls?

    Solution: Notice that every time we roll a die, our expected value is

    $$\frac{1+2+3+4+5+6}{6} = \frac{7}{2}.$$

    We can separate our calculations in two parts: the first roll and the remaining rolls. Since we have a $\frac{1}{6}$ of rolling each of 1-6, we can simply sum the expected values of the second part and then divide by 6. Since we are rolling 1, 2, 3, 4, 5, and 6 more times, the respective EVs are

    $$3.5, 3.5 \times 2, 3.5 \times 3, 3.5 \times 4, 3.5 \times 5, 3.5 \times 6,$$

    which sums to $(3.5)(21)$. The values of the first rolls are $1, 2, 3, 4, 5, 6$, intuitively. The sum of all of these values is $(4.5)(21)$ and so the expected value is $\frac{(4.5)(21)}{6} = \boxed{\frac{63}{4}}$.

    Proposed by Nithin Kavi

24. **Problem:** Dora has an extraneous calculator that can only perform 2 operations: either adding 1 to the current number or squaring the current number. Each minute, Dora randomly chooses an operation to apply to her number. She starts with 0. What is the expected number of minutes it takes Dora's number to become greater than or equal to 10?

    Solution: For each integer $0 \leq i \leq 9$, let E_i equal the expected number of minutes it takes to get to 10 starting from i. We are looking for E_0. We have $E_9 = 1$ since $9+1$ and 9^2 are both greater than or equal to 10. For each $4 \leq i \leq 8$, since $i^2 > 10$, we have

    $$E_i = \frac{1}{2}(E_{i+1}+1) + \frac{1}{2}(1) = \frac{1}{2}E_{i+1} + 1.$$

    We can then calculate

    $$E_8 = \frac{3}{2}, E_7 = \frac{7}{4}, E_6 = \frac{15}{8}, E_5 = \frac{31}{16}, E_4 = \frac{63}{32}.$$

    Then,

    $$E_3 = \frac{1}{2}(E_9+1) + \frac{1}{2}(E_4+1) = 1 + \frac{95}{64} = \frac{159}{64},$$

    and

    $$E_2 = \frac{1}{2}(E_4+1) + \frac{1}{2}(E_3+1) = \frac{413}{128}.$$

    We have

    $$E_1 = \frac{1}{2}(E_2+1) + \frac{1}{2}(E_1+1) \Rightarrow E_1 = E_2 + 2,$$

    and similarly $E_0 = E_1 + 2$, so $E_0 = E_2 + 4 = \frac{925}{128}$, for a final answer of $\boxed{1053}$.

    Proposed by Jerry Tan

25. **Problem:** Let $\triangle ABC$ be such that $AB = 2$, $BC = 1$, and $\angle ACB = 90^\circ$. Let points D and E be such that $\triangle ADE$ is equilateral, D is on segment $\overline{BC}$, and D and E are not on the same side of $\overline{AC}$. Segment $\overline{BE}$ intersects the circumcircle of $\triangle ADE$ at a second point F. If $BE = \sqrt{6}$, find the length of $\overline{BF}$.

Solution: First note that $\triangle ABC$ is a 30-60-90, meaning that $\angle ABC = 60^\circ$. Let $BD = x$. Extend BD past D to point G such that $\angle BGE = 60^\circ$. Let H be the intersection of BA and GE. Then $\triangle BGH$ is equilateral. Since $\triangle ADE$ is equilateral, we have by symmetry that $AB = DG = HE = 2$ and $BD = GE = AH = x$. Let I be the foot of the altitude from B to GH. Then $GI = \frac{x+2}{2}$, and

$$IE = GI - GE = \frac{x+2}{2} - x = \frac{2-x}{2},$$

and $BI = \sqrt{3}(\frac{x+2}{2})$. By the Pythagorean Theorem, $BI^2 + IE^2 = BE^2$, so

$$3\left(\frac{x^2+4x+4}{4}\right) + \frac{x^2-4x+4}{4} = 6$$

$$x^2 + 2x - 2 = 0 \Rightarrow \; x = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}.$$

Since x is positive, so $x = \sqrt{3} - 1$. Then, let $J \neq A$ be the intersection of the circumcircle of $\triangle ADE$ and AB. JDEA is cyclic, so $\angle AJD = 180 - \angle AED = 120$, then $\angle BJD = 180 - \angle AJD = 60$, so $\triangle BJD$ is equilateral. Thus $BJ = BD = \sqrt{3} - 1$. Finally, by power of a point, $BJ \cdot BA = BF \cdot BE \;\Rightarrow\; BF = 2(\sqrt{3}-1)/(\sqrt{6}) = \boxed{\dfrac{3\sqrt{2}-\sqrt{6}}{3}}$.

Proposed by Jerry Tan

Accuracy Round

1. **Problem:** Compute $45 \times 45 - 6$.

   Solution: We compute
   $$45 \times 45 - 6 = 2025 - 6 = \boxed{2019}$$

   Proposed by Allen Wang

2. **Problem:** Consecutive integers have nice properties. For example, $3, 4, 5$ are three consecutive integers, and $8, 9, 10$ are three consecutive integers also. If the sum of three consecutive integers is 24, what is the smallest of the three numbers?

   Solution: Notice that the sum of three consecutive integers is three times the middle integer. Therefore, the middle integer must be 8, and the smallest of the three consecutive numbers is $\boxed{7}$.

   Proposed by Allen Wang

3. **Problem:** How many positive integers less than 25 are either multiples of 2 or multiples of 3?

   Solution: We will first count the total multiples of 2 and total multiples of 3 separately, then subtract the total multiples of 6 which will be accounted for twice. There are a total of $\frac{24}{2} = 12$ multiples of 2, a total of $\frac{24}{3} = 8$ multiples of 3, and a total of $\frac{24}{6} = 4$ multiples of 6. The answer is $12 + 8 - 4 = \boxed{16}$.

   Proposed by Allen Wang

4. **Problem:** Charlotte has 5 positive integers. Charlotte tells you that the mean, median, and unique mode of his five numbers are all equal to 10. What is the largest possible value of the one of Charlotte's numbers?

   Solution: Optimally, we want two of the numbers to be 10 and two other numbers to be as small as possible to maximize the greatest number. The set of 4 numbers satisfying these conditions is $\{1, 2, 10, 10\}$. Since the mean is 10, the sum of the 5 numbers must be 50, the final number is $50 - 10 - 10 - 2 - 1 = \boxed{27}$, which is also the greatest possible value of an integer.

   Proposed by Allen Wang

5. **Problem:** Mr. Meeseeks starts with a single coin. Every day, Mr. Meeseeks goes to a magical coin converter where he can either exchange 1 coin for 5 coins or exchange 5 coins for 3 coins. What is the least number of days Mr. Meeseeks needs to end with 15 coins?

   Solution: We will look at the two exchange options as net changes. The first option will increase Mr. Meeseeks's coin total by 4 while the second option will decrease Mr. Meeseeks's coin total by 2. Since Mr. Meeseeks begins with 1 coin and needs to end with 15 coins, we need to find the least number of days in which a net change of 14 coins can be made. This occurs by selecting the first option 4 times and then selecting the second option once. Thus, it will take Mr. Meeseeks a minimum of $\boxed{5}$ days.

   Proposed by Akshay Gowrishankar

6. **Problem:** Twelve years ago, Violet's age was twice her sister Holo's age. In 7 years, Holo's age will be 13 more than a third of Violet's age. 3 years ago, Violet and Holo's cousin Rindo's age was the sum of their ages. How old is their cousin?

   Solution: Let Holo's current age be x and Violet's current age be y. Given the first condition, we have that
   $$2(x - 12) = y - 12$$
   or
   $$2x - y = 12$$

Also, by the second condition, we have that

$$x + 7 = \frac{1}{3}(y + 3) + 13$$

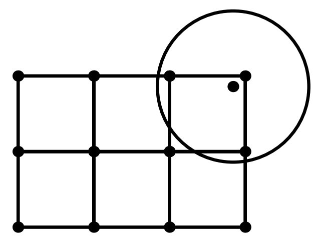
or

$$3x - y = 25$$

Subtracting the first equation from the second, we get that $x = 13$ and plugging x back into the first equation, we find that $y = 14$. Rindo's age 3 years ago was the sum of Violet and Holo's ages, or $(x-3) + (y-3)$ = 21. Thus, Rindo's current age is $\boxed{24}$.

Proposed by Akshay Gowrishankar and Justin Shan

7. **Problem:** In a 2 x 3 rectangle composed of 6 unit squares, let S be the set of all points P in the rectangle such that a unit circle centered at P covers some point in exactly 3 of the unit squares. Find the area of the region S. For example, the diagram below shows a valid unit circle in a 2 x 3 rectangle.



Solution: We realize that the only configuration by which a circle can intersect three of the unit squares is if it intersects three adjacent squares and is centered in one of the corner squares. To prevent the circle from intersect four different unit squares, we must make sure that it does not cross into the square located diagonally to the corner square, so therefore the center of our circle must be at most 1 unit away from the intersection of the four circles. We can now draw another circle of radius one centered at the intersection, and the region that we are looking for is what is inside the corner square but outside that circle.

This yields $1 - \frac{\pi}{4}$, and accounting for each of the possible corner squares in which we can center the circle, we have that the total area is $\boxed{4 - \pi}$

Proposed by Jerry Tan

8. **Problem:** What are the last four digits of 2^{1000}?

Solution: We are looking for $2^{1000} \pmod{10000}$. We use the Chinese Remainder Theorem, noting that $10000 = 2^4 \cdot 5^4 = 16 \cdot 625$. Clearly, $2^{1000} \equiv 0 \pmod{16}$.

Note that $\phi(625) = \frac{4}{5} \cdot 625 = 500$. Therefore, by Euler we have $2^{500} \equiv 1 \pmod{625}$, so $2^{1000} \equiv (2^{500})^2 \equiv 1^2 \equiv 1 \pmod{625}$.

Thus, we are looking for a number which is divisible by 16 and 1 more than a multiple of 625. This gives us the equation $16m = 625n + 1$. Taking this equation $\mod 16$, we get $n \equiv -1 \equiv 15 \pmod{16}$. Plugging in $n = 15$, we get our final answer of $15 \cdot 625 + 1 = \boxed{9376}$.

Proposed by Poonam Sahoo

9. **Problem:** There a point X in the center of a 2 x 2 x 2 box. Find the volume of the region closer to X than the vertices of the box.

Solution: We may split the box into 8 unit cubes. Each cube has one vertex that is the center of the original cube and one vertex that is an original vertex. These vertices are opposites on the unit cubes.

Therefore, the region closer to x in each unit cube is exactly $\frac{1}{2}$ the volume, which is $\frac{1}{2}$. Since we have 8 cubes, we have a volume of $8 \times \frac{1}{2} = \boxed{4}$.

Proposed by Richard Huang

10. **Problem:** Evaluate $\sqrt{37 \cdot 41 \cdot 113 \cdot 290 - 4319^2}$.

    Solution: Note that all primes which are $1 \pmod 4$ can be expressed as the sum of two squares. Then, we have

    $$37 = 6^2 + 1^2, 41 = 4^2 + 5^2,$$

    and

    $$113 = 7^2 + 8^2.$$

    Similarly, we let

    $$290 = 11^2 + 13^2.$$

    We also know that

    $$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2.$$

    Then, we have

    $$(6^2 + 1^2)(4^2 + 5^2) = (30 + 4)^2 + (24 - 5)^2 = 34^2 + 19^2.$$

    Then, we use

    $$(34^2 + 19^2)(7^2 + 8^2) = (272 + 133)^2 + (238 - 152)^2 = 405^2 + 86^2.$$

    Finally, we have

    $$(405^2 + 86^2)(11^2 + 13^2) = (405 \cdot 13 - 11 \cdot 86)^2 + (405 \cdot 11 + 86 \cdot 13) = 4319^2 + 5573^2.$$

    Therefore, our answer is simply

    $$\sqrt{4319^2 + 5573^2 - 4319^2} = \boxed{5573.}$$

Proposed by Nithin Kavi

11. **Problem:** **Estimation:** A number is *abundant* if the sum of all its divisors is greater than twice number. One such number is 12, because $1 + 2 + 3 + 4 + 6 + 12 = 28 > 24$. How many abundant positive integers less than 20190 are there?

    Solution: By the following Python script, the answer is $\boxed{5002}$.

```
import math

print("Input number")
n = int(input())
counter = 0

def divisor_sum(number):
        divisors = []
        for i in range (1, math.floor(number/2)+1):
                if number%i==0:
                        divisors.append(i)
        return sum(divisors)

for i in range (2,n+1):
        if divisor_sum(i) > i:
                counter += 1
print(counter)
```

Proposed by Nithin Kavi

Team Round

Round 1

1. **Problem:** Suppose a certain menu has 3 sandwiches and 5 drinks. How many ways are there to pick a meal so that you have exactly a drink and a sandwich?

   Solution: There are $3 \cdot 5 = \boxed{15}$ ways to pick a meal.

   Proposed by Akshay Gowrishankar

2. **Problem:** If $a + b = 4$ and $a + 3b = 222222$, find $10a + b$.

   Solution: Subtracting the first equation from the second equation, we have that $2b = 222218$, so $b = 111109$. Then $a = 4 - b = 4 - 111109 = -111105$. Thus, $10a + b = \boxed{-999941}$.

   Proposed by Antonio Frigo

3. **Problem:** Compute
$$\left\lfloor \frac{2019 \cdot 2017}{2018} \right\rfloor$$
   where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

   Solution: Note
$$\left\lfloor \frac{2019 \cdot 2017}{2018} \right\rfloor = \left\lfloor \frac{(2018+1) \cdot (2018-1)}{2018} \right\rfloor = \left\lfloor \frac{2018^2 - 1}{2018} \right\rfloor = \boxed{2017}.$$

   Proposed by Allen Wang

Round 2

1. **Problem:** Andrew has 10 water bottles, each of which can hold at most 10 cups of water. Three bottles are thirty percent filled, five are twenty-four percent filled, and the rest are empty. What is the average amount of water, in cups, contained in the ten water bottles?

   Solution: Three of the bottles have $0.3 \cdot 10 = 3$ cups, and five of the bottles have $0.24 \cdot 10 = 2.4$. Then we have a total of $3 \cdot 3 + 5 \cdot 2.4 = 21$, so the average amount of water is $\frac{21}{10} = \boxed{2.1}$ cups.

   Proposed by Aaron Zhang

2. **Problem:** How many positive integers divide 195 evenly?

   Solution: The prime factorization of 195 is $3 \cdot 5 \cdot 13$, so it has $(1+1)(1+1)(1+1) = \boxed{8}$ positive factors.

   Proposed by Aaron Zhang

3. **Problem:** Square A has side length ℓ and area 128. Square B has side length $\ell/2$. Find the length of the diagonal of Square B.

   Solution: Since $\ell^2 = 128$, we have $\ell = 8\sqrt{2}$. Then for square B, its side length is $4\sqrt{2}$, so its diagonal is $4\sqrt{2} \cdot \sqrt{2} = \boxed{8}$.

   Proposed by Aaron Zhang

Round 3

1. **Problem:** A right triangle with area 96 is inscribed in a circle. If all the side lengths are positive integers, what is the area of the circle? Express your answer in terms of π.

   Solution: The area of a triangle is $\frac{bh}{2}$. This means that $bh = 96 \times 2 = 192$. This gives the following possibilities for base and height, assuming that the height is larger than the base:

   $$(1, 192), (2, 96), (3, 64), (4, 48), (6, 32), (8, 24), (12, 16).$$

   Note that $12 - 16 - 20$ is a Pythagorean triple equivalent to $3 - 4 - 5$. Thus, the base and height are $(12, 16)$, so the hypotenuse is 20. This means the radius of the circle is 10, giving an area of 100π.

   Proposed by Aaron Zhang

2. **Problem:** A circular spinner has four regions labeled $3, 5, 6, 10$. The region labeled 3 is $1/3$ of the spinner, 5 is $1/6$ of the spinner, 6 is $1/10$ of the spinner, and the region labeled 10 is $2/5$ of the spinner. If the spinner is spun once randomly, what is the expected value of the number on which it lands?

   Solution: The expected value is

   $$3 \times \frac{1}{3} + 5 \times \frac{1}{6} + 6 \times \frac{1}{10} + 10 \times \frac{2}{5} = \boxed{\frac{193}{30}}$$

   Proposed by Nithin Kavi

3. **Problem:** Find the integer k such that $k^3 = 8353070389$.

   Solution: Notice that 8353070389 is slightly greater than 8×10^9. The cube root of 8×10^9 is 2000 so we can deduce that k should be slightly greater than 2000. Furthermore, the units digit of k must be 9, which is the only units digit that produces a 9 when cubed. Testing numbers beginning from 2009, it becomes obvious that 2009^3 and 2019^3 do not produce 8 as a tens digit. We find that $k = \boxed{2029}$ is the right answer.

   Proposed by Jerry Tan

Round 4

1. **Problem:** How many ways are there to arrange the letters in the word *zugzwang* such that the two z's are never consecutive?

   Solution: There are 8 letters, including two z's and two g's. Thus there are $\frac{8!}{2! \times 2!} = 10,080$ possible orderings. Now we must determine how many possibilities there are such that the two z's are consecutive. We do so by assuming the the two z's are one letter, a "double z" or zz. Now there are 7 letters, including two g's. Thus there are $\frac{7!}{2!} = 2,520$ possible orderings where the z's are consecutive. This implies there are $10,080 - 2,520 = \boxed{7,560}$ possibilities.

   Proposed by Nithin Kavi

2. **Problem:** If O is the circumcenter of $\triangle ABC$, $\overline{AD}$ is the altitude from A to $\overline{BC}$, $\angle CAB = 66^\circ$ and $\angle ABC = 44^\circ$, then what is the measure of $\angle OAD$?

   Solution: We have that $m\angle ACB = 180 - 44 - 66 = 70$, so $m\angle DAC = 90 - 70 = 20$. Then by the inscribed angle theorem, $m\angle AOC = 2 \cdot 44 = 88$. Then $m\angle OAC = (180 - 88)/2 == 46$, so $m\angle AOD = 46 - 20 = \boxed{26}$.

   Proposed by Aaron Zhang

3. **Problem:** If $x > 0$ satisfies $x^3 + \frac{1}{x^3} = 18$, find $x^5 + \frac{1}{x^5}$.

   Solution:

   Let $x + 1/x = r$. Then, we have $r^3 = x^3 + \frac{1}{x^3} + 3(x + 1/x) = 18 + 3r$. From $r^3 = 18 + 3r$, we note that $r = 3$ is a solution. Thus, this polynomial factors as $(r-3)(r^3 + 3r + 6)$. This means $r = 3$ since $r > 0$ is real.

   From $x + 1/x = 3$, we get that $x^2 + \frac{1}{x^2} = 3^2 - 2 = 7$ from squaring that equation. Then we have:

   $$\left(x^3 + \frac{1}{x^3}\right)\left(x^2 + \frac{1}{x^2}\right) = x^5 + \frac{1}{x^5} + x + 1/x = x^5 + \frac{1}{x^5} + 3.$$

   Our final answer is $18 \cdot 7 - 3 = \boxed{123}$.

   Proposed by Nithin Kavi

Round 5

1. **Problem:** Let C be the answer to Question 3. Neethen decides to run for school president! To be entered onto the ballot, however, Neethen needs $C + 1$ signatures. Since no one else will support him, Neethen gets the remaining C other signatures through bribery. The situation can be modeled by $k \cdot N = 495$, where k is the number of dollars he gives each person, and N is the number of signatures he will get. How many dollars does Neethen have to bribe each person with?

   Solution: N is the number of people Neethen bribes, we can say $N = C$, and our answer is $A = \frac{495}{C}$. As shown later (see problem 3 solution), $A = \boxed{11}$

   Proposed by Akshay Gowrishankar

2. **Problem:** Let A be the answer to Question 1. With $3A - 1$ total votes, Neethen still comes short in the election, losing to Serena by just 1 vote. Darn! Neethen sneaks into the ballot room, knowing that if he destroys just two ballots that voted for Serena, he will win the election. How many ways can Neethen choose two ballots to destroy?

   Solution: Serena got $3A$ votes, so the number of ways Neethen can choose ballots to destroy is $B = \binom{3A}{2}$. As shown later (see problem 3 solution), $A = 11$, so $B = \binom{33}{2} = \frac{33*32}{2} = \boxed{528}$.

   Proposed by Akshay Gowrishankar

3. **Problem:** Let B be the answer to Question 2. Oh no! Neethen is caught rigging the election by the principal! For his punishment, Neethen needs to run the perimeter of his school three times. The school is modeled by a square of side length k furlongs, with k being an integer. If Neethen runs B feet in total, what is $k + 1$.

   Solution: Neethen runs a distance of 12 side lengths, which means that B is a multiple of 12. From part 2, we have $B = 12 * k = \frac{(3A)*(3A-1)}{2}$. Thus, $(3A) * (3A - 1) \equiv 0 \mod 8$, meaning either $3A$ or $3A - 1$ is a multiple of 8. Since A is a factor of 495, A must be odd, so $3A - 1 \equiv 0 \mod 8$, which means that $A \equiv 3 \mod 8$. We now look at the factors of 495, and find that the only ones that are congruent to 3, 11, and 99. We can approximate B to be $\frac{9*A^2}{2}$, and thus we can estimate C to be $\frac{9*A^2}{24} \approx \frac{A^2}{3}$. Thus, since $C * A = 495$, we have $\frac{A^3}{3} \approx 495$, so $A^3 \approx 1500$ and $A \approx 12$. Out of our 3 possibilities, the only reasonable answer is $A = 11$. This means that $B = \binom{33}{2} = 528$ and $C = \frac{495}{11} = \boxed{45}$.

   Proposed by Akshay Gowrishankar

Round 6

1. **Problem:** Find the unique real positive solution to the equation $x = \sqrt{6 + 2\sqrt{6} + 2x} - \sqrt{6 - 2\sqrt{6} - 2x} - \sqrt{6}$.

   Solution:

   We begin by adding $\sqrt{6}$ to both sides followed by squaring. This gives us:

   $$x^2 + 2x\sqrt{6} + 6 = 6 + 2\sqrt{6} + 2x + 6 - 2\sqrt{6} - 2x + 2\sqrt{36 - (2x + 2\sqrt{6})^2} = 12 + 2\sqrt{36 - (2x + 2\sqrt{6})^2}.$$

   Then we get $x^2 + 2x\sqrt{6} - 6 = 2\sqrt{12 - 8x\sqrt{6} - 4x^2}$.

   Let $a = x^2 + 2x\sqrt{6}$. Then we have $a - 6 = 2\sqrt{12 - 4a}$.

   Squaring both sides, we get $a^2 - 12a + 36 = 48 - 16a$. Then $a^2 + 4a - 12 = (a + 6)(a - 2) = 0$, so $a = -6, 2$.

   If $a = -6$, then $x^2 + 2x\sqrt{6} + 6 = 0$, so $x = -\sqrt{6} < 0$. The problem requires a positive solution.

   Thus $a = 2$. Then $x^2 + 2x\sqrt{6} - 2 = 0$, so $x = \frac{-2\sqrt{6} \pm \sqrt{24+8}}{2} = \boxed{2\sqrt{2} - \sqrt{6}}$.

   Proposed by Jerry Tan

2. **Problem:** Consider triangle ABC with $AB = 13$ and $AC = 14$. Point D lies on BC, and the lengths of the perpendiculars from D to AB and AC are both $\frac{56}{9}$. Find the largest possible length of BD.

   Solution: Let E, F be the feet of the altitudes from D to AB, AC, respectively. The area of the triangle is

   $$(DE \cdot AB + DF \cdot AC)/2 = DE \cdot (AB + AC)/2 = \frac{56}{9}(27)/2 = 84.$$

   Let $BC = x$. By Heron's formula,

   $$84 = \sqrt{\frac{x+27}{2} \cdot \frac{x+1}{2} \cdot \frac{x-1}{2} \cdot \frac{27-x}{2}} = \frac{\sqrt{(x^2-1)(729-x^2)}}{4} = \frac{\sqrt{-x^4 + 730x^2 - 729}}{4}$$

   $$\Rightarrow\ x^4 - 730x^2 + 336^2 + 729 = 0.$$

   There are 2 ways to proceed: either calculate 336^2 and bash, or note that this quartic can be factored into the form $(x^2 - M)(x^2 - N) = 0$. Then, note from before that a 13-14-x triangle has area 84, so we can check that $x = 15$, meaning $x^2 - 225$, is root. Then $x^4 - 730x^2 + 336^2 + 729 = (x^2 - 225)(x^2 - N)$, so $N = 730 - 225 = 505$, so $BC = x = \sqrt{505}$. We check that this satisfies the triangle inequality and $\sqrt{505} > 15$.

   Then, since $AD = AD$ by the reflexive property, $DE = DF$, and $\angle AED = \angle AFD = 90$, we get $\triangle AED \cong \triangle AFD$ by HL congruence. Thus $\angle EAD = \angle FAD$, so DA is the angle bisector of $\angle BAC$. By the angle bisector theorem, $BD = \frac{13}{13+14} \cdot BC = \boxed{\frac{13\sqrt{505}}{27}}$.

   Proposed by Jerry Tan

3. **Problem:** Let $f(x, y) = \frac{m}{n}$, where m is the smallest positive integer such that x and y divide m, and n is the largest positive integer such that n divides both x and y. If $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, what is the median of the distinct values that $f(a, b)$ can take, where $a, b \in S$?

   Solution: If one of the positive integers is 1 and the other one varies from 1 to 10, we get the numbers from 1 to 10. When the largest number is at least 2, the only cases that produce numbers distinct from 1 to 10 inclusive are the ones where a, b are relatively prime. We get the following:

   $f(2,7) = 14, f(2,9) = 18.$

$f(3,4) = 12, f(3,5) = 15, f(3,7) = 21, f(3,8) = 24, f(3,10) = 30.$

$f(4,5) = 20, f(4,7) = 28, f(4,9) = 36.$

$f(5,7) = 35, f(5,8) = 40, f(5,9) = 45.$

$f(6,7) = 42.$

$f(7,8) = 56, f(7,9) = 63, f(7,10) = 70$

$f(8,9) = 72$

$f(9,10) = 90.$

There are 29 numbers, so the median is the $15th$ number. This is 1 through 10, 12, 14, 15, 18, $\boxed{20.}$

Proposed by Aaron Zhang

Round 7

1. **Problem:** The polynomial $y = x^4 - 22x^2 - 48x - 23$ can be written in the form

$$y = (x - \sqrt{a} - \sqrt{b} - \sqrt{c})(x - \sqrt{a} + \sqrt{b} + \sqrt{c})(x + \sqrt{a} - \sqrt{b} + \sqrt{c})(x + \sqrt{a} + \sqrt{b} - \sqrt{c})$$

for positive integers a, b, c with $a \le b \le c$. Find $(a + b) \cdot c$.

Solution: Multiplying the polynomial out using difference of squares, we get

$$\begin{aligned} y &= ((x - \sqrt{a})^2 - (\sqrt{b} + \sqrt{c})^2)((x + \sqrt{a})^2 - (\sqrt{b} - \sqrt{c})^2) \\ &= (x^2 - 2\sqrt{a}x + a - (b + c + 2\sqrt{bc}))(x^2 + 2\sqrt{a}x + a - (b + c - 2\sqrt{bc})) \\ &= (x^2 + a - b - c - (2\sqrt{a}x + 2\sqrt{bc}))(x^2 + a - b - c + (2\sqrt{a}x + 2\sqrt{bc})) \\ &= (x^2 + a - b - c)^2 - (2\sqrt{a}x + 2\sqrt{bc})^2 \\ &= x^4 + a^2 + b^2 + c^2 + 2ax^2 - 2bx^2 - 2cx^2 - 2ab - 2ac + 2bc - (4ax^2 + 4bc + 8x\sqrt{abc}) \\ &= x^4 - (2a + 2b + 2c)x^2 - 8\sqrt{abc}x + a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \end{aligned}$$

Now, we can match coefficients. We have

$$\begin{aligned} 2(a + b + c) = 22 &\Rightarrow\ a + b + c = 11 \\ 8\sqrt{abc} = 48 &\Rightarrow\ abc = 36 \\ a^2 + b^2 + c^2 - 2ab - 2bc - 2ca &= -23. \end{aligned}$$

Then

$$121 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca,$$

subtracting the third equation above from this, we get

$$121 - (-23) = 4(ab + bc + ca) \Rightarrow\ ab + bc + ca = 36.$$

Then by Vieta's, we get that a, b, c are roots to the cubic $x^3 - 11x^2 + 36x - 36 = 0$, and using rational root theorem we get $(x - 2)(x - 3)(x - 6) = 0$. Thus, $a = 2, b = 3, c = 6$, and the final answer is $(2 + 3) \cdot 6 = \boxed{30}$.

Proposed by Justin Shan

2. **Problem:** Varun is grounded for getting an F in every class. However, because his parents don't like him, rather than making him stay at home they toss him onto a number line at the number 3. A wall is placed at 0 and a door to freedom is placed at 10. To escape the number line, Varun must reach 10, at which point he walks through the door to freedom. Every 5 minutes a bell rings, and Varun may walk

to a different number, and he may not walk to a different number except when the bell rings. Being an F student, rather than walking straight to the door to freedom, whenever the bell rings Varun just randomly chooses an adjacent integer with equal chance and walks towards it. Whenever he is at 0 he walks to 1 with a 100 percent chance. What is the expected number of times Varun will visit 0 before he escapes through the door to freedom?

Solution: For each integer $0 \leq i \leq 10$, let e_i be the expected number of times he visits 0 before 10 starting from the number i. For each $2 \leq i \leq 9$, since there is a 50-50 chance he goes in either direction, we have $e_i = \frac{1}{2}(e_{i+1} + e_{i-1})$. Thus, $e_1, e_2, \ldots e_{10}$ forms an arithmetic sequence. We also have $e_{10} = 0$, $e_1 = \frac{1}{2}(e_0 + 1) + \frac{1}{2}e_2$, and $e_0 = e_1$. Let $e_9 = x$. Since we have an arithmetic sequence, $e_8 = 2x, e_7 = 3x, \ldots, e_0 = e_1 = 9x$. Finally, $9x = e_1 = \frac{1}{2}(e_0 + 1) + \frac{1}{2}e_2 = \frac{1}{2}(9x + 1) + \frac{1}{2}(8x) = 8.5x + 0.5 \Rightarrow\ 0.5x = 0.5 \Rightarrow\ x = 1$. Thus $e_3 = 7x = \boxed{7}$.

Proposed by Nithin Kavi

3. **Problem:** Let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of positive integers such that every element divides 36 under the condition that $a_1 < a_2 < \cdots < a_6$. Find the probability that one of these chosen sets also satisfies the condition that every $a_i \mid a_j$ if $i|j$.

   Solution: Notice first that $36 = 2^2 \cdot 3^2$ so it has $3 \times 3 = 0$ factors. The total number of ways to pick a sequence of 6 increasing integers out of a sequence of these 9 distinct factors is simply $\binom{9}{6} = 84$, which is the denominator. We can then find the number of sequences that satisfy the conditions through casework:
   Case 1a: We will split the case where $a_1 = 1$ into several cases. By the last condition, a_1 must divide every other number in the sequence which 1 satisfies for all sequences. We will examine the sub-case where $a_2 = 2$. After examining all possibilities where $a_3 = 3, 4, 6$ or 9, we find that there are 23 sequences that work.
   Case 1b: In this sub-case, we will examine all possible sequences when $a_1 = 1$ and $a_2 = 3$. Here, a_3 can equal $4, 6$ or 9. A quick check will find that 12 sequences fall in this case.
   Case 1c: Now, examine all possible sequences when $a_1 = 1$ and $a_2 = 4$. Here, a_3 can equal 6 or 9. A quick check will find that 2 sequences fall in this case.
   Case 1d: Now, examine all possible sequences when $a_1 = 1$ and $a_2 = 6$. Here, a_3 can only equal 9. A quick check reveals that only 1 sequence falls in this case.

   (Note: Clearly, a_2 cannot be greater than 6 because there are only 4 factors of 36 greater than 6. Therefore, we only have to check up to the case where $a_2 = 6$)

   Case 2: We will now check the case where $a_1 = 2$. Since a_1 must divide the rest of the numbers in the sequence, and since 36 only has 6 even factors. The only possible sequence is $\{2, 4, 6, 12, 18, 36\}$. This sequence does satisfy all the conditions in the problem, so 1 sequence falls in this case.

   Case 3: We will now check the case where $a_1 = 3$. There are also only 6 factors of 36 that are multiples of 3. By the same logic in case 2, we only need to check to see if this sequence, namely $\{3, 6, 9, 12, 18, 36\}$, satisfies all conditions in the original question, which it does. Therefore, 1 sequences falls in this case.

   We realize that a_1 must divide the 5 other terms in the sequence, so it can only be $1, 2,$ or 3. Therefore, we have examined all possible cases. The total number of sequences that work is $23 + 12 + 2 + 1 + 1 + 1 = 40$. Our answer is $\frac{40}{84} = \boxed{\frac{10}{21}}$.

   Proposed by Aaron Zhang

Round 8

1. **Problem:** How many numbers between 1 and $100,000$ can be expressed as the product of at most 3 distinct primes?

Your answer will be scored according to the following formula, where X is the correct answer and I is your input.

$$\max\left\{0, \left\lceil \min\left\{13 - \frac{|I-X|}{0.1|I|}, 13 - \frac{|I-X|}{0.1|I-2X|}\right\}\right\rceil\right\}.$$

Solution: By the following program, we find the answer is $\boxed{32905}$.

```
import math
import itertools

def findsubsets(S,m):
    return set(itertools.combinations(S, m))

def multiplyList(myList) :
    result = 1
    for x in myList:
         result = result * x
    return result


primes = [2]
for n in range(2,100000):
        for i in range(2,int(math.sqrt(n))+2):
                if n%i==0:
                        break
                elif i==int(math.sqrt(n))+1:
                        primes.append(n)
counter = 0
for i in range (1,3):
        print(i)
        for k in findsubsets(primes,i):
                if multiplyList(k)<= 100000:
                        counter += 1
print(counter)
```

Proposed by Antonio Frigo