# Acton-Boxborough Math Competition Online Contest Solutions

Saturday, October 20 — Sunday, October 21, 2018

1. **Problem:** Compute the greatest integer less than or equal to

$$\frac{10+12+14+16+18+20}{21}.$$

**Solution:** We see that this equals $\frac{3 \cdot 30}{21} = 4.28571$, so our answer is $\boxed{4}$.

*Proposed by Antonio Frigo*

2. **Problem:** Let $A = 1, B = 2, C = 3, \ldots, Z = 26$. Find $A + B + M + C$.

**Solution:** Clearly, $A = 1, B = 2$ and $C = 3$ as the problem states, while $M = 13$. The answer is $1 + 2 + 13 + 3 = \boxed{19}$.

*Proposed by Nithin Kavi*

3. **Problem:** In Mr. M's farm, there are 10 cows, 8 chicken, and 4 spiders. How many legs are there (including Mr. M's legs)?

**Solution:** Cows have 4 legs each, chickens have 2 legs each and spiders have 8 legs each. We must add 2 legs at the end to account for Mr. M himself. The answer is $10 \cdot 4 + 8 \cdot 2 + 4 \cdot 8 + 2 = 40 + 16 + 32 + 2 = \boxed{90}$.

*Proposed by Allen Wang*

4. **Problem:**The area of an equilateral triangle with perimeter 18 inches. is in the form $\frac{a\sqrt{b}}{c}$, where $a$ and $c$ are relatively prime and $b$ is not divisible by the square of any prime. Find $a + b + c$.

**Solution:** From the formula for the area of an equilateral triangle with side length $s$, we get an area of $\frac{6^2\sqrt{3}}{4} = \frac{9\sqrt{3}}{1}$, so $a + b + c = \boxed{13}$.

*Proposed by Antonio Frigo*

5. **Problem:** Let $f$ be a linear function so $f(x) = ax + b$ for some $a$ and $b$. If $f(1) = 2017$ and $f(2) = 2018$, what is $f(2019)$?

**Solution:** We know that $f(1) = a + b = 2017$, and $f(2) = 2a + b = 2018$. Solving this system yields $a = 1$ and $b = 2016$. So $f(2019) = 1(2019) + 2016 = \boxed{4035}$.

*Proposed by Allen Wang*

6. **Problem:** How many integers $m$ satisfy $4 < m^2 \leq 216$?

**Solution:** We get that $2 < |m| \leq 14 < \sqrt{216}$, so $m = \pm 3, \pm 4, \ldots, \pm 14$. We have $\boxed{24}$ solutions in all.

*Proposed by Nithin Kavi*

7. **Problem:** Allen and Michael Phelps compete at the Olympics for swimming. Allen swims $\frac{9}{8}$ the distance Phelps swims, but Allen swims in $\frac{5}{9}$ of Phelps's time. If Phelps swims at a rate of 3 kilometers per hour, what is Allen's rate of swimming? Express your answer as a common fraction in kilometers per hour.

**Solution:** We know that $\frac{\text{distance}}{\text{time}} = rate$. Phelp's rate is $\frac{3km}{1hr}$. If we set his time and distance that way, we can base Allen's time and distance as well, as rate is distance over time, so the Allen's rate will not be affected. Allen's time = $\frac{5}{9} \times 1 = \frac{5}{9}hr$. We have his distance = $\frac{9}{8} \times 3 = \frac{27}{8}km$. Therefore, Allen's rate is $\frac{27/8}{5/9} = \boxed{243/40}$ kilometers per hour. Our final answer is $\boxed{283}$.

*Proposed by Eddie Wang*

8. **Problem:** Let $X$ be the number of distinct arrangements of the letters in "POONAM," $Y$ be the number of distinct arrangements of the letters in "ALLEN" and $Z$ be the number of distinct arrangements of the letters in "NITHIN." Evaluate $\frac{X+Z}{Y}$.

   **Solution:** We have $X = \frac{6!}{2!} = 360, Y = \frac{5!}{2!} = 60$ and $Z = \frac{6!}{2!2!} = 180$. This gives us an answer of

$$\frac{360 + 180}{60} = \boxed{9}.$$

   *Proposed by Nithin Kavi*

9. **Problem:** Two overlapping circles, both of radius 9cm, have centers that are 9cm apart. The combined area of the two circles can be expressed as $\frac{a\pi + b\sqrt{c} + d}{e}$. Find $a + b + c + d + e$.

   **Solution:** To solve, we need to find the area of the intersection of the two circles. Consider one of the right triangles formed by one of the radii from a center to a point where the two circles intersect, half of the radical axis and half of the segment connecting the two circles. Since one of its bases is half the segment connecting the two centers, it has length $\frac{9}{2}$.

   We know its hypotenuse has length 9. Hence, this right triangle must be a 30-60-90 triangle, and the central angle formed by the two radii must be $2 \times 60^\circ = 120^\circ$. Now consider the sector of one of the circles formed by the $120^\circ$ central angle. We can find its area, subtract the area of the triangle formed by two of the radii and the radical axis, then double it to find the area of the intersection of the two circles. The area of the intersection is:

$$2 \cdot \left[ \frac{120^\circ}{360^\circ} \cdot (9^2 \cdot \pi) - \frac{1}{2} \cdot (\frac{9}{2}) \cdot (9\sqrt{3}) \right] = 2 \cdot (27\pi - \frac{81\sqrt{3}}{4}) = 54\pi - \frac{81\sqrt{3}}{2}$$

   Now we can subtract the area of the intersection from the sum of the areas of the two circles to find the combined area which is:

$$2 \cdot 81\pi - (54\pi - \frac{81\sqrt{3}}{2}) = 108\pi + \frac{81\sqrt{3}}{2} = \frac{216\pi + 81\sqrt{3}}{2}$$

   Our answer is

$$216 + 81 + 3 + 0 + 2 = \boxed{302}$$

   *Proposed by Antonio*

10. **Problem:** In the Boxborough-Acton Regional High School (BARHS), 99 people take Korean, 55 people take Maori, and 27 people take Pig Latin. 4 people take both Korean and Maori, 6 people take both Korean and Pig Latin, and 5 people take both Maori and Pig Latin. 1 especially ambitious person takes all three languages, and and 100 people do not take a language. If BARHS does not offer any other languages, how many students attend BARHS?

    **Solution:** We know in total, 99 take Korean, 55 take Maori, 27 take Pig Latin. However we overcount. By PIE we subtract the people taking two languages and add back the people who take three languages and so forth.

    Therefore, we have $99 + 55 + 27 - (4 + 6 + 5) + (1) = 167$. Additionally, we have 100 people who don't take any language, so we end up with $167 + 100 = \boxed{267}$ students at BARHS.

    *Proposed by Eddie Wang*

11. **Problem:**Let $H$ be a regular hexagon of side length 2. Let $M$ be the circumcircle of $H$ and $N$ be the inscribed circle of $H$. Let $m, n$ be the area of $M$ and $N$ respectively. The quantity $m - n$ is in the form $\pi a$, where $a$ is an integer. Find $a$.

    **Solution:** Note that the incircle and circumcircle of $H$ both share the same center, except the vertices of $H$ lie on the circumcircle and the incircle of $H$ is tangent to the sides of $H$ at their midpoints.

    So, the radius of the incircle is the distance from the center to the midpoint of one of the sides of $H$. This length is equivalent to the altitude of an equilateral triangle with sides of length 2, so the radius of the incircle is $\sqrt{3}$.

    Now, the radius of the circumcircle is the distance from the center to one of the vertices. Since the side length of $H$ is 2, the circumradius is also 2. Then the area of the incircle is $n = (\sqrt{3})^2\pi = 3\pi$, and the area of the circumcircle is $m = (2^2)\pi = 4\pi$, so

    $$m - n = 4\pi - 3\pi = \pi \cdot 1.$$

    Thus $a = \boxed{1}$.

    *Proposed by Antonio Frigo*

12. **Problem:** How many ordered quadruples of positive integers $(p, q, r, s)$ are there such that $p + q + r + s \leq 12$?

    **Solution:** When we substitute $p' = p - 1, q' = q - 1, r' = r - 1$, and $s' = s - 1$, the problem is equivalent to counting the number of ordered quadruples of nonnegative integers $(p', q', s', t')$ such that $p' + q' + r' + s' \leq 8$. However, this is equal to the number of quintuples of nonnegative integers $(p', q', r', s', t)$ such that $p' + q' + r' + s' + t = 8$; note that we see this is true since for any valid quintuple, the first four integers form a valid quadruple. Thus, the problem is reduced to a distribution problem of dividing 12 balls into 5 bins. The number of ways to distribute is equal to the number of ways to permute 8 balls and 4 bars, where the bars divide the balls into the corresponding distribution. For example, the permutation $OO|OOO||OOOOOO|O$ corresponds to the distribution $(2, 3, 0, 6, 1)$. The number of such permutations is $\binom{12}{4} = \boxed{495}$, the answer.

    *Proposed by Nithin Kavi*

13. **Problem:** Let $K = 2^{\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\left(1+\frac{1}{3^{16}}\right)\cdots}$. What is $K^8$?

    **Solution:** Let $P = \left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\left(1 + \frac{1}{3^{16}}\right)\ldots$. If we multiply both sides of this equation by $1 - \frac{1}{3^2}$, we get:

    $$\frac{8}{9}P = \left(1 - \frac{1}{3^2}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\ldots = \left(1 - \frac{1}{3^4}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{3^8}\right)\ldots.$$

    We see that the product telescopes to 1 through repeated difference of squares. Then $P = \frac{9}{8}$, so $K^8 = \left(2^{\frac{9}{8}}\right)^8 = \boxed{512}$.

    *Proposed by Nithin Kavi*

14. **Problem:** Neetin, Neeton, Neethan, Neethine, and Neekhil are playing basketball. Neetin starts out with the ball. How many ways can they pass 5 times so that Neethine ends up with the ball?

    **Solution:** We define $a(n)$ to be the number of ways to pass $n$ times to get the ball back to Neetin, and we define $b(n)$ to be the number of ways to pass $n$ times to get the ball to a specific person not Neetin (in both situations with Neetin beginning with the ball). We can create two recursions from $a(n)$ and

$b(n)$. If Neetin has the ball after $n$ steps, at the $n-1$th step another person must have had the ball, so $a(n) = 4b(n-1)$.

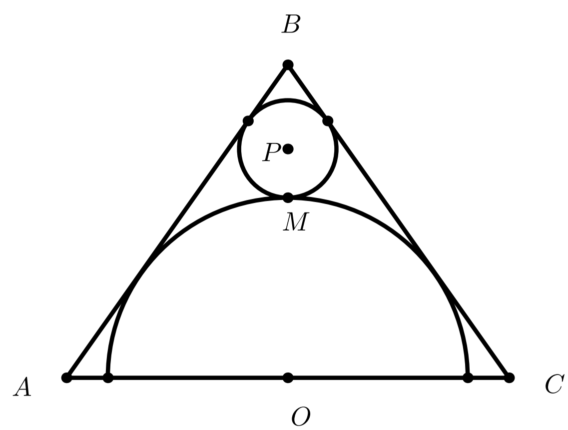
However, if Neetin does not have the ball after $n$ steps, at the $n-1$th step, the ball could have been with Neetin or someone else. If the ball was with Neetin, which can occur in $a(n-1)$ ways there is one way to pass the ball to the person who has the ball at step $n$, giving $a(n-1)$ ways. On the other hand, if the ball was with someone else not Neetin and not the person at step $n$, it could have been with 3 other people in $3b(n-1)$ ways. So $b(n) = a(n-1) + 3b(n-1)$.

Substituting gives $b(n) = 3b(n-1) + 4b(n-2)$, and we would like to compute $b(5)$. We see that our initial value takes $b(1) = 1$ and $b(2) = 3$. Then, $b(3) = 3 \cdot 3 + 4 \cdot 1 = 13$, $b(4) = 3 \cdot 13 + 4 \cdot 3 = 51$, $b(5) = 3 \cdot 51 + 4 \cdot 13 = \boxed{205}$.

*Proposed by Antonio Frigo*

15. **Problem:** In an octahedron with side lengths 3, inscribe a sphere. Then inscribe a second sphere tangent to the first sphere and to 4 faces of the octahedron. The radius of the second sphere can be expressed in the form $\frac{\sqrt{a}-\sqrt{b}}{c}$, where the square of any prime factor of $c$ does not evenly divide into $b$. Compute $a+b+c$.

**Solution:**




The diagram above is obtained by taking a cross section through the apex, and two midpoints of opposite edges of the square. From this, it is evident that $AB = BC = \frac{3\sqrt{3}}{2}$ and $AC = 3$. Then $AO = \frac{3}{2}$ and $BO = \frac{3\sqrt{2}}{2}$ by the Pythagorean Theorem. Call the point where $\overline{AB}$ is tangent to the semicircle point $E$ (note that it is a semicircle rather than a whole circle because this diagram is only of the top half of the octahedron).

Then from the area of $\triangle ABO$, we get that $AO \cdot BO = AB \cdot OE \Rightarrow \frac{3}{2} \cdot \frac{3\sqrt{2}}{2} = \frac{3\sqrt{3}}{2} \cdot OE \Rightarrow OE = \frac{\sqrt{6}}{2}$.

Let $F$ be the point where $\overline{AB}$ is tangent to the smaller circle. It is clear that $\triangle BFI \sim \triangle BEO$. Let $r$ be the radius of circle $I$, which is the value that we are trying to find. Then $BI = \frac{3\sqrt{2}}{2} - \frac{\sqrt{6}}{2} - r$. Plugging this into similar triangle ratios, we get:

$$\frac{r}{\frac{\sqrt{6}}{2}} = \frac{\frac{3\sqrt{2}}{2} - \frac{\sqrt{6}}{2} - r}{\frac{3\sqrt{2}}{2}}$$

$$\frac{2r}{\sqrt{6}} = \frac{3 - \sqrt{3} - r\sqrt{2}}{3}$$

$$6r = 3\sqrt{6} - 3\sqrt{2} - 2\sqrt{3} \cdot r$$

$$r = \frac{3\sqrt{6}-3\sqrt{2}}{6+2\sqrt{3}} \cdot \frac{6-2\sqrt{3}}{6-2\sqrt{3}} = \frac{18\sqrt{6}-6\sqrt{18}-18\sqrt{2}+6\sqrt{6}}{24} = \frac{2\sqrt{6}-3\sqrt{2}}{2} = \frac{\sqrt{24}-\sqrt{18}}{2}.$$

The final answer is $24+18+2 = \boxed{44}$.

*Proposed by Nithin Kavi*